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Supplementary Appendix for Online Publication

Panel Data and Experimental Design

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Appendix: For online publication

A Derivations and proofs

We are the first to derive analytic power calculation formulas for panel data models under non-i.i.d. error structures. In this section, we derive power calculation analytics for cross-sectional, difference-in-differences, and collapsed data estimators. We present the resulting equations in Sections 2 and 3 of the main text.

We first re-derive well-known power calculation formulas, for both a cross-sectional experiment and a panel experiment that applies the difference-in-differences estimator, under the assumption that error terms are uncorrelated. We then relax this assumption to consider the difference-in-differences estimator applied to a panel experiment in which the error structure of the data exhibits an arbitrary form of serial correlation. We show that the previously reported power calculation formula is incorrect in this case, and we derive the first power calculation formula that correctly incorporates arbitrary serial correlation in a panel data setting. We then consider a collapsed data research design, and we again show that the previously reported power calculation formula is incorrect in the presence of serial correlation, whereas our new power calculation formula gives the correct analytic results.

We then provide the proofs to the lemmas presented in the main text. These lemmas show that the above power calculation formulas can be applied even in the presence of cross-sectional correlations, so long as treatment is randomly assigned at the unit level. In particular, Lemma 2 states that the variance estimator we derive, which accounts for serial correlation, gives an unbiased estimate of the true variance under unit-level randomization, even when cross-sectional correlations exist. In other words, our newly derived power calculation formula can be applied to any panel experiment setting, regardless of the true error structure of the data, so long as treatment is randomly assigned to units.

*The main text is available at: `https://ei.haas.berkeley.edu/research/papers/WP277.pdf`

A.1 Cross section

A.1.1 Independent error structure

Model There are J units randomly assigned a treatment status D_i, with proportion P in treatment ($D_i = 1$) and proportion $(1 - P)$ in control ($D_i = 0$). The units are indexed so $i \in [1, PJ]$ is treated and $j \in [PJ + 1, J]$ is a control. We make standard assumptions for randomized trials:

Assumption 1 (Data generating process)**.** *The data are generated according to the following model:*

$$Y_i = \beta + \tau D_i + \varepsilon_i \tag{A1}$$

where Y_i *is the outcome of interest for unit* i*;* β *is the expected outcome of non-treated units;* τ *is the treatment effect which is homogenous across all units;* D_i *is a treatment indicator; and* ε_i *is an idiosyncratic error term distributed i.i.d.* $\mathcal{N}(0, \sigma_\varepsilon^2)$*.*

Assumption 2 (Strict exogeneity)**.** $\mathrm{E}[\varepsilon_i \mid \mathbf{X}] = 0$*, where* $\mathbf{X} = [\beta \ D]$*. In practice, this follows from random assignment of* D_i*.*

Coefficient estimate The coefficient estimates from an OLS regression are

$$\begin{aligned}
\begin{pmatrix} \widehat{\beta} \\ \widehat{\tau} \end{pmatrix} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'Y \\
&= \begin{pmatrix} J & PJ \\ PJ & PJ \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^J Y_i \\ \sum_{i=1}^J D_i Y_i \end{pmatrix} \\
&= \frac{1}{P(1-P)J^2} \begin{pmatrix} PJ\left(\sum_{i=1}^J Y_i - \sum_{i=1}^J D_i Y_i\right) \\ J\left(\sum_{i=1}^J D_i Y_i - P\sum_{i=1}^J Y_i\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{(1-P)J} \sum_{i=1}^J (1 - D_i) Y_i \\ \frac{1}{PJ} \sum_{i=1}^J D_i Y_i - \frac{1}{(1-P)J} \sum_{i=1}^J (1 - D_i) Y_i \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{(1-P)J} \sum_{i=PJ+1}^J Y_i \\ \frac{1}{PJ} \sum_{i=1}^{PJ} Y_i - \frac{1}{(1-P)J} \sum_{i=PJ+1}^J Y_i \end{pmatrix}
\end{aligned}$$

Defining the mean outcome in the treatment and control groups, respectively, as

$$\begin{aligned}
\overline{Y}_T &= \frac{1}{PJ} \sum_{i=1}^{PJ} Y_i \\
\overline{Y}_C &= \frac{1}{(1-P)J} \sum_{i=PJ+1}^{J} Y_i
\end{aligned}$$

gives coefficient estimates of

$$\begin{aligned}
\widehat{\beta} &= \overline{Y}_C \\
\widehat{\tau} &= \overline{Y}_T - \overline{Y}_C
\end{aligned}$$

Variance of coefficient estimate The variance of the estimate of the treatment effect, $\widehat{\tau}$, is

$$\begin{aligned}\text{Var}\left(\widehat{\tau} \mid \mathbf{X}\right) &= \text{Var}\left(\overline{Y}_T \mid \mathbf{X}\right) + \text{Var}\left(\overline{Y}_C \mid \mathbf{X}\right) - 2\,\text{Cov}\left(\overline{Y}_T, \overline{Y}_C \mid \mathbf{X}\right) \\ &= \text{Var}\left(\overline{Y}_T \mid \mathbf{X}\right) + \text{Var}\left(\overline{Y}_C \mid \mathbf{X}\right) \end{aligned} \tag{A2}$$

The first term of Equation (A2) is

$$\begin{aligned}\text{Var}\left(\overline{Y}_T \mid \mathbf{X}\right) &= \text{Var}\left(\frac{1}{PJ}\sum_{i=1}^{PJ} Y_i \mid \mathbf{X}\right) \\ &= \frac{1}{(PJ)^2}\sum_{i=1}^{PJ} \text{Var}(Y_i \mid \mathbf{X}) \\ &= \frac{\sigma_\varepsilon^2}{PJ}\end{aligned} \tag{A3}$$

Similarly, the second term of Equation (A2) is

$$\text{Var}\left(\overline{Y}_C \mid \mathbf{X}\right) = \frac{\sigma_\varepsilon^2}{(1-P)J} \tag{A4}$$

Substituting Equations (A3) and (A4) into Equation (A2) gives

$$\begin{aligned}\text{Var}(\widehat{\tau} \mid \mathbf{X}) &= \frac{\sigma_\varepsilon^2}{PJ} + \frac{\sigma_\varepsilon^2}{(1-P)J} \\ &= \frac{\sigma_\varepsilon^2}{P(1-P)J}\end{aligned} \tag{A5}$$

This is equal to the variance estimator produced by an OLS regression of Equation (A1).

Minimum detectable effect The minimum detectable effect (MDE), or the smallest treatment effect we have the power to detect, is

$$\begin{aligned}MDE &= \left(t_{1-\kappa}^{J-2} + t_{\alpha/2}^{J-2}\right)\sqrt{\text{Var}\left(\widehat{\tau} \mid \mathbf{X}\right)} \\ &= \left(t_{1-\kappa}^{J-2} + t_{\alpha/2}^{J-2}\right)\sqrt{\frac{\sigma_\varepsilon^2}{P(1-P)J}}\end{aligned} \tag{A6}$$

where κ is the power of the hypothesis test, α is the significance level, and the critical values are drawn from t-distributions with $J-2$ degrees of freedom. We present this well-known result as Equation (2) in the main text.

A.2 Difference-in-differences

A.2.1 Independent error structure

Model In this model, P proportion of the J units are again randomized into treatment. The researcher collects the outcome Y_{it} for each unit i, across m pre-treatment time periods and r post-treatment time periods. For units in the treatment group, $D_{it} = 0$ in pre-treatment periods and $D_{it} = 1$ in post-treatment periods; for units in the control group, $D_{it} = 0$ in all $(m + r)$ periods.

Assumption 3 (Data generating process). *The data are generated according to the following model:*

$$\begin{aligned} Y_{it} &= \beta + \tau D_{it} + \varepsilon_{it} \\ &= \beta + \tau D_{it} + \upsilon_i + \delta_t + \omega_{it} \end{aligned} \tag{A7}$$

where Y_{it} is the outcome of interest for unit i at time t; β is the expected outcome of non-treated observations; τ is the treatment effect that is homogenous across all units and all time periods; D_{it} is a time-varying treatment indicator; υ_i is a time-invariant unit effect distributed i.i.d. $\mathcal{N}(0, \sigma^2_\upsilon)$; δ_t is a common time effect distributed i.i.d. $\mathcal{N}(0, \sigma^2_\delta)$; and ω_{it} is an idiosyncratic error term distributed i.i.d. $\mathcal{N}(0, \sigma^2_\omega)$.

Assumption 4 (Strict exogeneity). $\mathrm{E}[\omega_{it} \mid \mathbf{X}] = 0$, *where $\mathbf{X}$ is a full rank matrix of regressors, including a constant, the treatment indicator D, $J - 1$ unit fixed effects, and $(m + r) - 1$ time fixed effects. This again follows from random assignment of D_{it}.*

Assumption 5 (Balanced panel). *The number of pre-treatment observations, m, and post-treatment observations, r, is the same for each unit, and all units are observed in every time period.*

Coefficient estimate The treatment effect, τ, can be estimated by OLS with unit and time fixed effects. In a balanced panel, this is equivalent to de-meaning at both the unit and time levels. Define

$$\ddot{Y}_{it} = Y_{it} - \overline{Y}_i - \overline{Y}_t + \overline{\overline{Y}} \tag{A8}$$

$$\ddot{D}_{it} = D_{it} - \overline{D}_i - \overline{D}_t + \overline{\overline{D}} \tag{A9}$$

$$\ddot{\omega}_{it} = \omega_{it} - \overline{\omega}_i - \overline{\omega}_t + \overline{\overline{\omega}} \tag{A10}$$

where

$$\begin{aligned} \overline{Y}_i &= \frac{1}{m+r} \sum_{t=-m+1}^{r} Y_{it} \\ \overline{Y}_t &= \frac{1}{J} \sum_{i=1}^{J} Y_{it} \\ \overline{\overline{Y}} &= \frac{1}{J(m+r)} \sum_{t=-m+1}^{r} \sum_{i=1}^{J} Y_{it} \end{aligned}$$

with $\overline{D}_i$, $\overline{D}_t$, $\overline{\overline{D}}$, $\overline{\omega}_i$, $\overline{\omega}_t$, and $\overline{\overline{\omega}}$ defined analogously. Substituting Equations (A8)–(A10) into Equation (A7) and simplifying gives the de-meaned DGP,

$$\ddot{Y}_{it} = \tau \ddot{D}_{it} + \ddot{\omega}_{it}$$

The estimate of the treatment effect is

$$\begin{aligned}
\widehat{\tau} &= (\ddot{D}'\ddot{D})^{-1}\ddot{D}'\ddot{Y} \\
&= \left(\sum_{i=1}^{J} \sum_{t=-m+1}^{r} \ddot{D}_{it}^2 \right)^{-1} \sum_{i=1}^{J} \sum_{t=-m+1}^{r} \ddot{D}_{it}\ddot{Y}_{it} \\
&= \frac{m+r}{P(1-P)Jmr} \left[\sum_{i=1}^{J} \sum_{t=-m+1}^{r} \ddot{D}_{it} Y_{it} - \sum_{i=1}^{J} \overline{Y}_i \sum_{t=-m+1}^{r} \ddot{D}_{it} \right. \\
&\qquad \left. - \sum_{t=-m+1}^{r} \overline{Y}_t \sum_{i=1}^{J} \ddot{D}_{it} + \overline{Y} \sum_{i=1}^{J} \sum_{t=-m+1}^{r} \ddot{D}_{it} \right] \\
&= \frac{m+r}{P(1-P)Jmr} \sum_{i=1}^{J} \sum_{t=-m+1}^{r} \ddot{D}_{it} Y_{it} \\
&= \frac{m+r}{P(1-P)Jmr} \left[\sum_{i=1}^{PJ} \left(\frac{-(1-P)r}{m+r} \sum_{t=-m+1}^{0} Y_{it} + \frac{(1-P)m}{m+r} \sum_{t=1}^{r} Y_{it} \right) \right. \\
&\qquad \left. + \sum_{i=PJ+1}^{J} \left(\frac{Pr}{m+r} \sum_{t=-m+1}^{0} Y_{it} + \frac{-Pm}{m+r} \sum_{t=1}^{r} Y_{it} \right) \right] \\
&= \frac{1}{PJ} \sum_{i=1}^{PJ} \left[\frac{-1}{m} \sum_{t=-m+1}^{0} Y_{it} + \frac{1}{r} \sum_{t=1}^{r} Y_{it} \right] - \frac{1}{(1-P)J} \sum_{i=PJ+1}^{J} \left[\frac{-1}{m} \sum_{t=-m+1}^{0} Y_{it} + \frac{1}{r} \sum_{t=1}^{r} Y_{it} \right] \\
&= \left(\overline{Y}_T^A - \overline{Y}_T^B \right) - \left(\overline{Y}_C^A - \overline{Y}_C^B \right)
\end{aligned}$$

where

$$\begin{aligned}
\overline{Y}_T^A &= \frac{1}{PJr} \sum_{i=1}^{PJ} \sum_{t=1}^{r} Y_{it} \\
\overline{Y}_T^B &= \frac{1}{PJm} \sum_{i=1}^{PJ} \sum_{t=-m+1}^{0} Y_{it} \\
\overline{Y}_C^A &= \frac{1}{(1-P)Jr} \sum_{i=PJ+1}^{J} \sum_{t=1}^{r} Y_{it} \\
\overline{Y}_C^B &= \frac{1}{(1-P)Jm} \sum_{i=PJ+1}^{J} \sum_{t=-m+1}^{0} Y_{it}
\end{aligned}$$

Variance of coefficient estimate The variance of the estimate of the treatment effect, $\widehat{\tau}$, is

$$\begin{aligned}
\operatorname{Var}(\widehat{\tau} \mid \mathbf{X}) &= \operatorname{Var}\left(\overline{Y}_T^A \mid \mathbf{X}\right) + \operatorname{Var}\left(\overline{Y}_T^B \mid \mathbf{X}\right) + \operatorname{Var}\left(\overline{Y}_C^A \mid \mathbf{X}\right) + \operatorname{Var}\left(\overline{Y}_C^B \mid \mathbf{X}\right) \\
&\quad - 2\operatorname{Cov}\left(\overline{Y}_T^A, \overline{Y}_T^B \mid \mathbf{X}\right) - 2\operatorname{Cov}\left(\overline{Y}_T^A, \overline{Y}_C^A \mid \mathbf{X}\right) + 2\operatorname{Cov}\left(\overline{Y}_T^A, \overline{Y}_C^B \mid \mathbf{X}\right) \\
&\quad + 2\operatorname{Cov}\left(\overline{Y}_T^B, \overline{Y}_C^A \mid \mathbf{X}\right) - 2\operatorname{Cov}\left(\overline{Y}_T^B, \overline{Y}_C^B \mid \mathbf{X}\right) - 2\operatorname{Cov}\left(\overline{Y}_C^A, \overline{Y}_C^B \mid \mathbf{X}\right) \\
&= \operatorname{Var}\left(\overline{Y}_T^A \mid \mathbf{X}\right) + \operatorname{Var}\left(\overline{Y}_T^B \mid \mathbf{X}\right) + \operatorname{Var}\left(\overline{Y}_C^A \mid \mathbf{X}\right) + \operatorname{Var}\left(\overline{Y}_C^B \mid \mathbf{X}\right) \\
&\quad - 2\Bigg[\operatorname{Cov}\left(\overline{Y}_T^A, \overline{Y}_T^B \mid \mathbf{X}\right) + \operatorname{Cov}\left(\overline{Y}_T^A, \overline{Y}_C^A \mid \mathbf{X}\right) \\
&\qquad + \operatorname{Cov}\left(\overline{Y}_T^B, \overline{Y}_C^B \mid \mathbf{X}\right) + \operatorname{Cov}\left(\overline{Y}_C^A, \overline{Y}_C^B \mid \mathbf{X}\right)\Bigg] \qquad \text{(A11)}
\end{aligned}$$

The first term of Equation (A11) is

$$\begin{aligned}
\operatorname{Var}\left(\overline{Y}_T^A \mid \mathbf{X}\right) &= \operatorname{Var}\left(\frac{1}{PJr}\sum_{i=1}^{PJ}\sum_{t=1}^{r} Y_{it}\right) \\
&= \frac{1}{(PJr)^2}\operatorname{Var}\left(\sum_{i=1}^{PJ}\sum_{t=1}^{r} Y_{it} \mid \mathbf{X}\right) \\
&= \frac{1}{PJr}\left[\operatorname{Var}(Y_{it} \mid \mathbf{X}) + (r-1)\operatorname{Cov}(Y_{it}, Y_{is} \mid \mathbf{X}) + (PJ-1)\operatorname{Cov}(Y_{it}, Y_{jt} \mid \mathbf{X})\right] \\
&= \frac{1}{PJr}\left(r\sigma_\upsilon^2 + PJ\sigma_\delta^2 + \sigma_\omega^2\right) \qquad \text{(A12)}
\end{aligned}$$

Similarly, the remaining variance terms of Equation (A11) are

$$\operatorname{Var}\left(\overline{Y}_T^B \mid \mathbf{X}\right) = \frac{1}{PJm}\left(m\sigma_\upsilon^2 + PJ\sigma_\delta^2 + \sigma_\omega^2\right) \qquad \text{(A13)}$$

$$\operatorname{Var}\left(\overline{Y}_C^A \mid \mathbf{X}\right) = \frac{1}{(1-P)Jr}\left(r\sigma_\upsilon^2 + (1-P)J\sigma_\delta^2 + \sigma_\omega^2\right) \qquad \text{(A14)}$$

$$\operatorname{Var}\left(\overline{Y}_C^B \mid \mathbf{X}\right) = \frac{1}{(1-P)Jm}\left(m\sigma_\upsilon^2 + (1-P)J\sigma_\delta^2 + \sigma_\omega^2\right) \qquad \text{(A15)}$$

The first covariance component of Equation (A11) is

$$\begin{aligned}
\operatorname{Cov}\left(\overline{Y}_T^A, \overline{Y}_T^B \mid \mathbf{X}\right) &= \operatorname{E}\left[\overline{Y}_T^A \overline{Y}_T^B \mid \mathbf{X}\right] - \operatorname{E}\left[\overline{Y}_T^A \mid \mathbf{X}\right]\operatorname{E}\left[\overline{Y}_T^B \mid \mathbf{X}\right] \\
&= \operatorname{E}\Bigg[\left(\frac{1}{PJr}\sum_{i=1}^{PJ}\sum_{t=1}^{r}(\beta + \tau + \upsilon_i + \delta_t + \omega_{it})\right) \\
&\qquad \times \left(\frac{1}{PJm}\sum_{i=1}^{PJ}\sum_{t=-m+1}^{0}(\beta + \upsilon_i + \delta_t + \omega_{it})\right) \mid \mathbf{X}\Bigg] \\
&\quad - \operatorname{E}\left[\frac{1}{PJr}\sum_{i=1}^{PJ}\sum_{t=1}^{r}(\beta + \tau + \upsilon_i + \delta_t + \omega_{it}) \mid \mathbf{X}\right]
\end{aligned}$$

$$
\begin{aligned}
&\times \mathrm{E}\left[\frac{1}{PJm}\sum_{i=1}^{PJ}\sum_{t=-m+1}^{0}(\beta+\upsilon_i+\delta_t+\ \omega_{it}) \mid \mathbf{X}\right] \\
&= \left[\beta(\beta+\tau)+\frac{1}{PJ}\mathrm{E}\left[\upsilon_i^2 \mid \mathbf{X}\right]\right]-\beta(\beta+\tau) \\
&= \frac{1}{PJ}\left(\mathrm{Var}(\upsilon_i)-\mathrm{E}\left[\upsilon_i \mid \mathbf{X}\right]^2\right) \\
&= \frac{\sigma_\upsilon^2}{PJ} \qquad \text{(A16)}
\end{aligned}
$$

Similarly, the remaining covariance terms of Equation (A11) are

$$
\mathrm{Cov}\left(\overline{Y}_T^A, \overline{Y}_C^A \mid \mathbf{X}\right) = \frac{\sigma_\delta^2}{r} \qquad \text{(A17)}
$$

$$
\mathrm{Cov}\left(\overline{Y}_T^B, \overline{Y}_C^B \mid \mathbf{X}\right) = \frac{\sigma_\delta^2}{m} \qquad \text{(A18)}
$$

$$
\mathrm{Cov}\left(\overline{Y}_C^A, \overline{Y}_C^B \mid \mathbf{X}\right) = \frac{\sigma_\upsilon^2}{(1-P)J} \qquad \text{(A19)}
$$

Substituting Equations (A12)–(A19) into Equation (A11) gives

$$
\begin{aligned}
\mathrm{Var}\left(\widehat{\tau} \mid \mathbf{X}\right) &= 2\frac{\sigma_\upsilon^2}{P(1-P)J}+2\frac{(m+r)\sigma_\delta^2}{mr}+\frac{(m+r)\sigma_\omega^2}{P(1-P)Jmr}-2\left[\frac{\sigma_\upsilon^2}{P(1-P)J}+\frac{(m+r)\sigma_\delta^2}{mr}\right] \\
&= \left(\frac{\sigma_\omega^2}{P(1-P)J}\right)\left(\frac{m+r}{mr}\right) \qquad \text{(A20)}
\end{aligned}
$$

This is equal to the variance estimator produced by an OLS regression of Equation (A7), except that it is scaled by σ_ω^2 instead of σ_ε^2.

Minimum detectable effect The MDE is

$$
\begin{aligned}
MDE &= \left(t_{1-\kappa}^J+t_{\alpha/2}^J\right)\sqrt{\mathrm{Var}\left(\widehat{\tau} \mid \mathbf{X}\right)} \\
&= \left(t_{1-\kappa}^J+t_{\alpha/2}^J\right)\sqrt{\left(\frac{\sigma_\omega^2}{P(1-P)J}\right)\left(\frac{m+r}{mr}\right)}
\end{aligned}
$$

This is the standard Frison and Pocock (1992) result, also referenced by McKenzie (2012), and is shown as Equation (3) in the main text. We assume J degrees of freedom to be consistent with the assumptions of the CRVE; alternatively, $J(m+r)-(J+m+r)$ degrees of freedom would be consistent with the assumptions of OLS standard errors. Note that this model is, in fact, an extension of the result in these papers, as both Frison and Pocock (1992) and McKenzie (2012) assume $\sigma_\delta^2 = 0$.

A.2.2 Serially correlated error structure

Model There are J units, P proportion of which are randomized into treatment. The researcher again collects outcome data Y_{it} for each unit i, across m pre-treatment time periods and r post-treatment time periods. For treated units, $D_{it} = 0$ in pre-treatment periods, and $D_{it} = 1$ in post-treatment periods; for control units, $D_{it} = 0$ in all periods.

Assumption 6 (Data generating process). *The data are generated according to the following model:*

$$Y_{it} = \beta + \tau D_{it} + \upsilon_i + \delta_t + \omega_{it} \tag{A21}$$

where Y_{it} is the outcome of interest for unit i at time t; β is the expected outcome of non-treated observations; τ is the treatment effect that is homogenous across all units and all time periods; D_{it} is a time-varying treatment indicator; υ_i is a unit-specific disturbance distributed i.i.d. $\mathcal{N}(0, \sigma_\upsilon^2)$; δ_t is a time-specific disturbance distributed i.i.d. $\mathcal{N}(0, \sigma_\delta^2)$; and ω_{it} is an idiosyncratic error term distributed (not necessarily i.i.d.) $\mathcal{N}(0, \sigma_\omega^2)$.

Assumption 7 (Strict exogeneity). $\mathrm{E}[\omega_{it} \mid \mathbf{X}] = 0$, *where $\mathbf{X}$ is a full rank matrix of regressors, including a constant, the treatment indicator D, $J-1$ unit fixed effects, and $(m+r)-1$ time fixed effects. This again follows from random assignment of D_{it}.*

Assumption 8 (Balanced panel). *The number of pre-treatment observations, m, and post-treatment observations, r, is the same for each unit, and all units are observed in every time period.*

Assumption 9 (Independence across units). $\mathrm{E}[\omega_{it}\omega_{js} \mid \mathbf{X}] = 0, \ \forall\, i \neq j, \ \forall\, t, s.$

Assumption 10 (Symmetric covariance structures). *Define:*

$$\psi_T^B \equiv \frac{2}{PJm(m-1)} \sum_{i=1}^{PJ} \sum_{t=-m+1}^{-1} \sum_{s=t+1}^{0} \mathrm{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right)$$

$$\psi_T^A \equiv \frac{2}{PJr(r-1)} \sum_{i=1}^{PJ} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \mathrm{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right)$$

$$\psi_T^X \equiv \frac{1}{PJmr} \sum_{i=1}^{PJ} \sum_{t=-m+1}^{0} \sum_{s=1}^{r} \mathrm{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right)$$

to be the average pre-treatment, post-treatment, and across-period covariance between different error terms of the same treated unit, respectively. Define ψ_C^B, ψ_C^A, and ψ_C^X analogously, where we consider the $(1-P)J$ control units instead of the PJ treated units. Using these definitions, assume that $\psi^B = \psi_T^B = \psi_C^B$; $\psi^A = \psi_T^A = \psi_C^A$; and $\psi^X = \psi_T^X = \psi_C^X$.[1]

Coefficient estimate The deterministic portion of this model is the same as that in Equation (A7), so the estimate of the treatment effect is again

$$\widehat{\tau} = \left(\overline{Y}_T^A - \overline{Y}_T^B\right) - \left(\overline{Y}_C^A - \overline{Y}_C^B\right)$$

1. We choose the letters "B" to indicate the Before-treatment period, and "A" to indicate the After-treatment period. We index the m pre-treatment periods $\{-m+1, \ldots, 0\}$, and the r post-treatment periods $\{1, \ldots, r\}$. In a randomized setting, $\mathrm{E}[\psi_T^B] = \mathrm{E}[\psi_C^B]$, $\mathrm{E}[\psi_T^A] = \mathrm{E}[\psi_C^A]$, and $\mathrm{E}[\psi_T^X] = \mathrm{E}[\psi_C^X]$, making this a reasonable assumption *ex ante*. However, it is possible for treatment to alter the covariance structure of treated units only.

Variance of coefficient estimate With a serially correlated error structure, Equation (A11) is still correct because of independence between observations that do not correspond to the same unit, but Equations (A12)–(A19) no longer hold. With serially correlated errors, the first term of Equation (A11) is

$$\begin{aligned}
\operatorname{Var}\left(\overline{Y}_T^A \mid \mathbf{X}\right) &= \operatorname{Var}\left(\frac{1}{PJr}\sum_{i=1}^{PJ}\sum_{t=1}^{r} Y_{it} \mid \mathbf{X}\right) \\
&= \frac{1}{(PJr)^2}\operatorname{Var}\left(\sum_{i=1}^{PJ}\sum_{t=1}^{r} Y_{it} \mid \mathbf{X}\right) \\
&= \frac{1}{(PJr)^2}\left[PJr\operatorname{Var}(Y_{it} \mid \mathbf{X}) + (PJ-1)r\operatorname{Cov}(Y_{it}, Y_{jt} \mid \mathbf{X}) + 2\sum_{i=1}^{PJ}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\operatorname{Cov}(Y_{it}, Y_{is} \mid \mathbf{X})\right] \\
&= \frac{\sigma_\varepsilon^2}{PJr} + \frac{(PJ-1)\sigma_\delta^2}{(PJ)^2 r} + \frac{2}{(PJr)^2}\sum_{i=1}^{PJ}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\left(\sigma_\upsilon^2 + \operatorname{Cov}(\omega_{it}, \omega_{is} \mid \mathbf{X})\right) \\
&= \frac{1}{PJr}\left(r\sigma_\upsilon^2 + PJ\sigma_\delta^2 + \sigma_\omega^2\right) + \frac{2}{(PJr)^2}\sum_{i=1}^{PJ}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\operatorname{Cov}(\omega_{it}, \omega_{is} \mid \mathbf{X}) \\
&= \frac{1}{PJr}\left(r\sigma_\upsilon^2 + PJ\sigma_\delta^2 + \sigma_\omega^2 + (r-1)\psi_T^A\right) \qquad \text{(A22)}
\end{aligned}$$

Similarly, with serial correlation, the remaining variance terms of Equation (A11) are

$$\operatorname{Var}\left(\overline{Y}_T^B \mid \mathbf{X}\right) = \frac{1}{PJm}\left(m\sigma_\upsilon^2 + PJ\sigma_\delta^2 + \sigma_\omega^2 + (m-1)\psi_T^B\right) \qquad \text{(A23)}$$

$$\operatorname{Var}\left(\overline{Y}_C^A \mid \mathbf{X}\right) = \frac{1}{(1-P)Jr}\left(r\sigma_\upsilon^2 + (1-P)J\sigma_\delta^2 + \sigma_\omega^2 + (r-1)\psi_C^A\right) \qquad \text{(A24)}$$

$$\operatorname{Var}\left(\overline{Y}_C^B \mid \mathbf{X}\right) = \frac{1}{(1-P)Jm}\left(m\sigma_\upsilon^2 + (1-P)J\sigma_\delta^2 + \sigma_\omega^2 + (m-1)\psi_C^B\right) \qquad \text{(A25)}$$

With serial correlation, the first covariance component of Equation (A11) is

$$\begin{aligned}
\operatorname{Cov}\left(\overline{Y}_T^A, \overline{Y}_T^B \mid \mathbf{X}\right) &= \operatorname{E}\left[\overline{Y}_T^A\overline{Y}_T^B \mid \mathbf{X}\right] - \operatorname{E}\left[\overline{Y}_T^A \mid \mathbf{X}\right]\operatorname{E}\left[\overline{Y}_T^B \mid \mathbf{X}\right] \\
&= \left[\beta(\beta+\tau) + \frac{1}{PJ}\operatorname{E}\left[\upsilon_i^2 \mid \mathbf{X}\right] + \frac{1}{(PJ)^2 mr}\sum_{i=1}^{PJ}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\operatorname{E}[\omega_{it}\omega_{is} \mid \mathbf{X}]\right] - \beta(\beta+\tau) \\
&= \frac{\sigma_\upsilon^2}{PJ} + \frac{1}{(PJ)^2 mr}\sum_{i=1}^{PJ}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\operatorname{Cov}(\omega_{it}, \omega_{is} \mid \mathbf{X}) \\
&= \frac{1}{PJ}\left(\sigma_\upsilon^2 + \psi_T^X\right) \qquad \text{(A26)}
\end{aligned}$$

Similarly, with serial correlation, the remaining covariance terms of Equation (A11) are

$$\operatorname{Cov}\left(\overline{Y}_T^A, \overline{Y}_C^A \mid \mathbf{X}\right) = \frac{\sigma_\delta^2}{r} \qquad \text{(A27)}$$

$$\text{Cov}\left(\overline{Y}_T^B, \overline{Y}_C^B \mid \mathbf{X}\right) = \frac{\sigma_\delta^2}{m} \tag{A28}$$

$$\text{Cov}\left(\overline{Y}_C^A, \overline{Y}_C^B \mid \mathbf{X}\right) = \frac{1}{(1-P)J}\left(\sigma_v^2 + \psi_C^X\right) \tag{A29}$$

Substituting Equations (A22)–(A29) into Equation (A11) and simplifying gives

$$\begin{aligned}
\text{Var}(\widehat{\tau} \mid \mathbf{X}) = & \frac{1}{PJr}\left(r\sigma_v^2 + PJ\sigma_\delta^2 + \sigma_\omega^2 + (r-1)\psi_T^A\right) \\
& + \frac{1}{PJm}\left(m\sigma_v^2 + PJ\sigma_\delta^2 + \sigma_\omega^2 + (m-1)\psi_T^B\right) \\
& + \frac{1}{(1-P)Jr}\left(r\sigma_v^2 + (1-P)J\sigma_\delta^2 + \sigma_\omega^2 + (r-1)\psi_C^A\right) \\
& + \frac{1}{(1-P)Jm}\left(m\sigma_v^2 + (1-P)J\sigma_\delta^2 + \sigma_\omega^2 + (m-1)\psi_C^B\right) \\
& - 2\left[\frac{1}{PJ}\left(\sigma_v^2 + \psi_T^X\right) + \frac{\sigma_\delta^2}{r} + \frac{\sigma_\delta^2}{m} + \frac{1}{(1-P)J}\left(\sigma_v^2 + \psi_C^X\right)\right] \\
= & \left(\frac{m+r}{P(1-P)Jmr}\right)\sigma_\omega^2 + \left(\frac{m-1}{PJm}\right)\psi_T^B + \left(\frac{r-1}{PJr}\right)\psi_T^A + \left(\frac{m-1}{(1-P)Jm}\right)\psi_C^B \\
& + \left(\frac{r-1}{(1-P)Jr}\right)\psi_C^A - \frac{2}{PJ}\psi_T^X - \frac{2}{(1-P)J}\psi_C^X
\end{aligned} \tag{A30}$$

To further simplify, we set $\psi^B = \psi_T^B = \psi_C^B$, $\psi^A = \psi_T^A = \psi_C^A$, and $\psi^X = \psi_T^X = \psi_C^X$. This follows from Assumption 10 above, whereby the average covariance term of the treated group is equal to the comparable covariance term of the control group.[2]

Then the variance of the estimate of the treatment effect is

$$\text{Var}(\widehat{\tau} \mid \mathbf{X}) = \left(\frac{1}{P(1-P)J}\right)\left[\left(\frac{m+r}{mr}\right)\sigma_\omega^2 + \left(\frac{m-1}{m}\right)\psi^B + \left(\frac{r-1}{r}\right)\psi^A - 2\psi^X\right] \tag{A31}$$

Neither Equation (A30) nor Equation (A31) is equal to Equation (A20), the variance of $\widehat{\tau}$ that is estimated by an OLS regression. Instead, the serially correlated error structure alters the true variance of the estimator such that the OLS estimator of the variance is not correct. The cluster-robust variance estimator must be used for correct inference.

Minimum detectable effect With serial correlation, the MDE is

$$\begin{aligned}
MDE = & \left(t_{1-\kappa}^J + t_{\alpha/2}^J\right)\sqrt{\text{Var}(\widehat{\tau} \mid \mathbf{X})} \\
= & \left(t_{1-\kappa}^J + t_{\alpha/2}^J\right)\left[\left(\frac{m+r}{P(1-P)Jmr}\right)\sigma_\omega^2 + \left(\frac{m-1}{PJm}\right)\psi_T^B + \left(\frac{r-1}{PJr}\right)\psi_T^A\right. \\
& + \left(\frac{m-1}{(1-P)Jm}\right)\psi_C^B + \left(\frac{r-1}{(1-P)Jr}\right)\psi_C^A \\
& \left. -\frac{2}{PJ}\psi_T^X - \frac{2}{(1-P)J}\psi_C^X\right]^{1/2}
\end{aligned}$$

2. Under random assignment into treatment, this will hold in expectation, as shown in Section A.3.

Following the same assumption as in Equation (A31), the MDE simplifies to

$$MDE = \left(t^J_{1-\kappa} + t^J_{\alpha/2}\right) \sqrt{\left(\frac{1}{P(1-P)J}\right) \left[\left(\frac{m+r}{mr}\right) \sigma^2_\omega + \left(\frac{m-1}{m}\right) \psi^B + \left(\frac{r-1}{r}\right) \psi^A - 2\psi^X\right]} \tag{A32}$$

where

$$\psi^B \equiv \frac{2}{Jm(m-1)} \sum_{i=1}^{J} \sum_{t=-m+1}^{-1} \sum_{s=t+1}^{0} \text{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right) \tag{A33}$$

$$\psi^A \equiv \frac{2}{Jr(r-1)} \sum_{i=1}^{J} \sum_{t=0}^{r-1} \sum_{s=t+1}^{r} \text{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right) \tag{A34}$$

$$\psi^X \equiv \frac{1}{Jmr} \sum_{i=1}^{J} \sum_{t=-m+1}^{0} \sum_{s=1}^{r} \text{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right) \tag{A35}$$

This is the serial-correlation-robust (SCR) power calculation formula, found in Equation (5) in the main text. Note that ψ^B, ψ^A, and ψ^X are likely to depend on the length of the pre- and post-treatment periods, m and r; serial correlation often diminishes as time periods become further apart, so larger values of m and r will result in less correlation on average and smaller parameter values. These parameters do not depend in a systematic way on the number of experimental units, J, however.

A.2.3 Serially and cross-sectionally correlated error structure

While randomization at the unit level eliminates cross-sectional correlation from Equation (A31) in expectation, we can also characterize the full variance without this random assignment assumption. This highlights the type of correlation structures that someone wanting to perform a power calculation for a quasi-experimental design might face.

Model There are J units, P proportion of which are randomized into treatment. The researcher again collects outcome data Y_{it} for each unit i, across m pre-treatment time periods and r post-treatment time periods. For treated units, $D_{it} = 0$ in pre-treatment periods, and $D_{it} = 1$ in post-treatment periods; for control units, $D_{it} = 0$ in all periods.

Assumption 11 (Data generating process)**.** *The data are generated according to the following model:*

$$Y_{it} = \beta + \tau D_{it} + \upsilon_i + \delta_t + \omega_{it}$$

where Y_{it} is the outcome of interest for unit i at time t; β is the expected outcome of non-treated observations; τ is the treatment effect that is homogenous across all units and all time periods; D_{it} is a time-varying treatment indicator; υ_i is a unit-specific disturbance distributed i.i.d. $\mathcal{N}(0, \sigma^2_\upsilon)$; δ_t is a time-specific disturbance distributed i.i.d. $\mathcal{N}(0, \sigma^2_\delta)$; and ω_{it} is an idiosyncratic error term distributed (not necessarily i.i.d.) $\mathcal{N}(0, \sigma^2_\omega)$.

Assumption 12 (Strict exogeneity)**.** $\mathrm{E}[\omega_{it} \mid \mathbf{X}] = 0$*, where* $\mathbf{X}$ *is a full rank matrix of regressors, including a constant, the treatment indicator* D*,* $J-1$ *unit fixed effects, and* $(m+r)-1$ *time fixed effects. This again follows from random assignment of* D_{it}*.*

Assumption 13 (Balanced panel)**.** *The number of pre-treatment observations,* m*, and post-treatment observations,* r*, is the same for each unit, and all units are observed in every time period.*

Assumption 14 (Independence across units at different times)**.** $\mathrm{E}[\omega_{it}\omega_{js} \mid \mathbf{X}] = 0, \ \forall \ i \neq j, \ t \neq s.$

Define ψ_i to be the average serial correlation parameters previously defined in Section A.2.2, with the subscript i denoting the correlation is within unit i. Also define ψ_t to be the comparable parameters characterizing cross-sectional correlations, with the subscript t denoting the correlation is within time t. For example, the average cross-sectional covariance among the treated group post-treatment is

$$\psi_{t,T}^{A} = \frac{2}{PJ(PJ-1)r} \sum_{i=1}^{PJ-1} \sum_{j=i+1}^{PJ} \sum_{t=1}^{r} \mathrm{Cov}(\omega_{it}, \omega_{jt} \mid \mathbf{X})$$

Coefficient estimate The deterministic portion of this model is the same as that in Equation (A7), so the estimate of the treatment effect is again

$$\widehat{\tau} = \left(\overline{Y}_T^A - \overline{Y}_T^B\right) - \left(\overline{Y}_C^A - \overline{Y}_C^B\right)$$

Variance of coefficient estimate As with the serially correlated error structure in Section A.2.2, Equation (A11) is still correct because of independence between observations that do not correspond to the same unit or time period, but Equations (A12)–(A19) no longer hold. With these arbitrary correlations, the first term of Equation (A11) is

$$\begin{aligned}
\mathrm{Var}\left(\overline{Y}_T^A \mid \mathbf{X}\right) &= \mathrm{Var}\left(\frac{1}{PJr}\sum_{i=1}^{PJ}\sum_{t=1}^{r} Y_{it} \mid \mathbf{X}\right) \\
&= \frac{1}{(PJr)^2}\mathrm{Var}\left(\sum_{i=1}^{PJ}\sum_{t=1}^{r} Y_{it} \mid \mathbf{X}\right) \\
&= \frac{1}{(PJr)^2}\Bigg[PJr\,\mathrm{Var}(Y_{it} \mid \mathbf{X}) + 2\sum_{i=1}^{PJ}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}(Y_{it}, Y_{is} \mid \mathbf{X}) \qquad \text{(A36)} \\
&\quad + 2\sum_{t=1}^{r}\sum_{i=1}^{PJ-1}\sum_{j=i+1}^{PJ}\mathrm{Cov}(Y_{it}, Y_{jt} \mid \mathbf{X})\Bigg] \\
&= \frac{\sigma_\varepsilon^2}{PJr} + \frac{2}{(PJr)^2}\sum_{i=1}^{PJ}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\left(\sigma_v^2 + \mathrm{Cov}(\omega_{it}, \omega_{is} \mid \mathbf{X})\right) \\
&\quad + \frac{2}{(PJr)^2}\sum_{t=1}^{r}\sum_{i=1}^{PJ-1}\sum_{j=i+1}^{PJ}\left(\sigma_\delta^2 + \mathrm{Cov}(\omega_{it}, \omega_{jt} \mid \mathbf{X})\right) \\
&= \frac{1}{PJr}\left(r\sigma_v^2 + PJ\sigma_\delta^2 + \sigma_\omega^2\right) + \frac{2}{(PJr)^2}\sum_{i=1}^{PJ}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}(\omega_{it}, \omega_{is} \mid \mathbf{X})
\end{aligned}$$

$$+\frac{2}{(PJr)^2}\sum_{t=1}^{r}\sum_{i=1}^{PJ-1}\sum_{j=i+1}^{PJ}\mathrm{Cov}(\omega_{it},\omega_{jt}\mid \mathbf{X})$$
$$=\frac{1}{PJr}\left(r\sigma_\upsilon^2+PJ\sigma_\delta^2+\sigma_\omega^2+(r-1)\psi_{i,T}^A+(PJ-1)\psi_{t,T}^A\right) \tag{A37}$$

Similarly, with arbitrary correlations, the remaining variance terms of Equation (A11) are

$$\mathrm{Var}\left(\overline{Y}_T^B\mid \mathbf{X}\right)=\frac{1}{PJm}\left(m\sigma_\upsilon^2+PJ\sigma_\delta^2+\sigma_\omega^2+(m-1)\psi_{i,T}^B+(PJ-1)\psi_{t,T}^B\right) \tag{A38}$$
$$\mathrm{Var}\left(\overline{Y}_C^A\mid \mathbf{X}\right)=\frac{1}{(1-P)Jr}\left(r\sigma_\upsilon^2+(1-P)J\sigma_\delta^2+\sigma_\omega^2+(r-1)\psi_{i,C}^A+((1-P)J-1)\psi_{t,C}^A\right) \tag{A39}$$
$$\mathrm{Var}\left(\overline{Y}_C^A\mid \mathbf{X}\right)=\frac{1}{(1-P)Jm}\left(m\sigma_\upsilon^2+(1-P)J\sigma_\delta^2+\sigma_\omega^2+(m-1)\psi_{i,C}^B+((1-P)J-1)\psi_{t,C}^B\right) \tag{A40}$$

With arbitrary correlations, the first covariance component of Equation (A11) is

$$\mathrm{Cov}\left(\overline{Y}_T^A,\overline{Y}_T^B\mid \mathbf{X}\right)=\mathrm{E}\left[\overline{Y}_T^A\overline{Y}_T^B\mid \mathbf{X}\right]-\mathrm{E}\left[\overline{Y}_T^A\mid \mathbf{X}\right]\mathrm{E}\left[\overline{Y}_T^B\mid \mathbf{X}\right]$$
$$=\left[\beta(\beta+\tau)+\frac{1}{PJ}\mathrm{E}\left[\upsilon_i^2\mid \mathbf{X}\right]+\frac{1}{(PJ)^2mr}\sum_{i=1}^{PJ}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{E}[\omega_{it}\omega_{is}\mid \mathbf{X}]\right]-\beta(\beta+\tau)$$
$$=\frac{\sigma_\upsilon^2}{PJ}+\frac{1}{(PJ)^2mr}\sum_{i=1}^{PJ}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}(\omega_{it},\omega_{is}\mid \mathbf{X})$$
$$=\frac{1}{PJ}\left(\sigma_\upsilon^2+\psi_{i,T}^X\right) \tag{A41}$$

Similarly, with arbitrary correlations, the remaining covariance terms of Equation (A11) are

$$\mathrm{Cov}\left(\overline{Y}_T^A,\overline{Y}_C^A\mid \mathbf{X}\right)=\frac{1}{r}\left(\sigma_\delta^2+\psi_{t,X}^A\right) \tag{A42}$$
$$\mathrm{Cov}\left(\overline{Y}_T^B,\overline{Y}_C^B\mid \mathbf{X}\right)=\frac{1}{m}\left(\sigma_\delta^2+\psi_{t,X}^B\right) \tag{A43}$$
$$\mathrm{Cov}\left(\overline{Y}_C^A,\overline{Y}_C^B\mid \mathbf{X}\right)=\frac{1}{(1-P)J}\left(\sigma_\upsilon^2+\psi_{i,C}^X\right) \tag{A44}$$

Substituting Equations (A37)–(A44) into Equation (A11) and simplifying gives

$$\mathrm{Var}\left(\widehat{\tau}\mid \mathbf{X}\right)=\frac{1}{PJr}\left(r\sigma_\upsilon^2+PJ\sigma_\delta^2+\sigma_\omega^2+(r-1)\psi_{i,T}^A+(PJ-1)\psi_{t,T}^A\right)$$
$$+\frac{1}{PJm}\left(m\sigma_\upsilon^2+PJ\sigma_\delta^2+\sigma_\omega^2+(m-1)\psi_{i,T}^B+(PJ-1)\psi_{t,T}^B\right)$$
$$+\frac{1}{(1-P)Jr}\left(r\sigma_\upsilon^2+(1-P)J\sigma_\delta^2+\sigma_\omega^2+(r-1)\psi_{i,C}^A+((1-P)J-1)\psi_{t,C}^A\right)$$
$$+\frac{1}{(1-P)Jm}\left(m\sigma_\upsilon^2+(1-P)J\sigma_\delta^2+\sigma_\omega^2+(m-1)\psi_{i,C}^B+((1-P)J-1)\psi_{t,C}^B\right)$$
$$-2\left[\frac{1}{PJ}\left(\sigma_\upsilon^2+\psi_{i,T}^X\right)+\frac{1}{r}\left(\sigma_\delta^2+\psi_{t,X}^A\right)+\frac{1}{m}\left(\sigma_\delta^2+\psi_{t,X}^B\right)+\frac{1}{(1-P)J}\left(\sigma_\upsilon^2+\psi_{i,C}^X\right)\right]$$

$$
\begin{aligned}
&= \left(\frac{m+r}{P(1-P)Jmr}\right)\sigma_\omega^2 + \left(\frac{m-1}{PJm}\right)\psi_{i,T}^B + \left(\frac{PJ-1}{PJm}\right)\psi_{t,T}^B \\
&+ \left(\frac{r-1}{PJr}\right)\psi_{i,T}^A + \left(\frac{PJ-1}{PJr}\right)\psi_{t,T}^A + \left(\frac{m-1}{(1-P)Jm}\right)\psi_{i,C}^B \\
&+ \left(\frac{(1-P)J-1}{(1-P)Jm}\right)\psi_{t,C}^B + \left(\frac{r-1}{(1-P)Jr}\right)\psi_{i,C}^A + \left(\frac{(1-P)J-1}{(1-P)Jr}\right)\psi_{t,C}^A \\
&- \frac{2}{PJ}\psi_{i,T}^X - \frac{2}{m}\psi_{t,X}^B - \frac{2}{r}\psi_{t,X}^A - \frac{2}{(1-P)J}\psi_{i,C}^X
\end{aligned}
$$

Minimum detectable effect With arbitrary correlations, the MDE is

$$
\begin{aligned}
MDE &= \left(t_{1-\kappa}^J + t_{\alpha/2}^J\right)\sqrt{\mathrm{Var}\left(\widehat{\tau} \mid \mathbf{X}\right)} \\
&= \left(t_{1-\kappa}^J + t_{\alpha/2}^J\right)\Bigg[\left(\frac{m+r}{P(1-P)Jmr}\right)\sigma_\omega^2 + \left(\frac{m-1}{PJm}\right)\psi_{i,T}^B + \left(\frac{PJ-1}{PJm}\right)\psi_{t,T}^B \\
&\qquad + \left(\frac{r-1}{PJr}\right)\psi_{i,T}^A + \left(\frac{PJ-1}{PJr}\right)\psi_{t,T}^A + \left(\frac{m-1}{(1-P)Jm}\right)\psi_{i,C}^B \\
&\qquad + \left(\frac{(1-P)J-1}{(1-P)Jm}\right)\psi_{t,C}^B + \left(\frac{r-1}{(1-P)Jr}\right)\psi_{i,C}^A + \left(\frac{(1-P)J-1}{(1-P)Jr}\right)\psi_{t,C}^A \\
&\qquad - \frac{2}{PJ}\psi_{i,T}^X - \frac{2}{m}\psi_{t,X}^B - \frac{2}{r}\psi_{t,X}^A - \frac{2}{(1-P)J}\psi_{i,C}^X\Bigg]^{1/2}
\end{aligned}
\tag{A45}
$$

We show in Section A.3 that Equation (A45) collapses (in expectation) to the serial-correlation-robust power calculation formula when treatment is randomly assigned.

A.2.4 Collapsed dataset

Bertrand, Duflo, and Mullainathan (2004) (henceforth BDM) suggest an alternative to the CRVE in order to achieve the correct false rejection rates in the presence of serial correlation: ignore the time-series structure of the data by averaging the pre-treatment data and the post-treatment data for each unit, then estimate a panel DD regression on this two-period collapsed dataset and apply the OLS variance estimator. While this does yield the desired false rejection rate, simply applying the FP formula to a collapsed dataset will *not* yield the desired power.

Consider again the model presented in Section A.2.2 under Assumptions 6–10, but now consider the case in which, prior to estimation, the data are collapsed to one pre- and one post-treatment observation per unit (to eliminate serial correlation, as suggested in BDM). The resulting DGP for the collapsed dataset is

$$
Y_{ip}^C = \beta + \tau D_{ip}^C + \upsilon_i + \delta_p^C + \omega_{ip}^C \tag{A46}
$$

where Y_{ip}^C is the average outcome for unit i for collapsed period p and the other variables are as defined in Section A.2.2 or are the collapsed analogs. Note that the τ in Equation (A46) is equivalent to that in Equation (A7). These models will yield the same estimate of the treatment effect, $\widehat{\tau}$, but different estimates of its variance in the presence of a (pre-collapsed) serially correlated error structure.

Equivalence of first-difference model BDM show that applying the OLS variance estimator to Equation (A46) achieves the correct false rejection rate. To see why this is the case, note that this collapsed model can alternatively be expressed as a first-difference model by subtracting each unit's collapsed pre-treatment data from its collapsed post-treatment data. Let ΔY_i^C be this difference for the outcome of interest, which gives

$$\begin{aligned}\Delta Y_i^C &= Y_{iA}^C - Y_{iB}^C \\ &= \left(\beta + \tau D_{iA}^C + \upsilon_i + \delta_A^C + \omega_{iA}^C\right) - \left(\beta + \tau D_{iB}^C + \upsilon_i + \delta_B^C + \omega_{iB}^C\right) \\ &= \tau\left(D_{iA}^C - D_{iB}^C\right) + \left(\delta_A^C - \delta_B^C\right) + \left(\omega_{iA}^C - \omega_{iB}^C\right)\end{aligned}$$

Defining the other differences variables similarly gives the first-difference DGP of

$$\Delta Y_i^C = \tau \Delta D_i^C + \Delta\delta^C + \Delta\omega_i^C \tag{A47}$$

Equations (A46) and (A47) are equivalent, so estimating these models yields not only the same estimate of the treatment effect, $\widehat{\tau}$, but also the same estimate of its variance. Note that the first-difference model, Equation (A47), is cross-sectional, so the error terms are i.i.d.[3] and the model meets the assumptions of OLS. As a result, the OLS variance estimator is unbiased for the first-difference model, as well as the equivalent collapsed model of Equation (A46).

Power Although using a collapsed dataset yields the correct false rejection rate, experiments will not be correctly powered if the FP formula is applied to a collapsed dataset. To see this, first consider the MDE of an experiment based on the first-difference model of Equation (A47). This is a cross-sectional model, so applying Equation (A6) yields:

$$\operatorname{Var}(\hat{\tau} \mid \mathbf{X}) = \frac{\sigma^2_{\Delta\omega^C}}{P(1-P)J} \tag{A48}$$

$$MDE = \left(t_{1-\kappa}^{J-2} + t_{\alpha/2}^{J-2}\right)\sqrt{\frac{\sigma^2_{\Delta\omega^C}}{P(1-P)J}} \tag{A49}$$

where $\sigma^2_{\Delta\omega^C}$ is the variance of the error term in the collapsed, first-difference model. This variance can be expressed as a function of the parameters that define the error structure of the collapsed data

$$\begin{aligned}\sigma^2_{\Delta\omega^C} &= \operatorname{Var}\left(\omega_{iA}^C - \omega_{iB}^C \mid \mathbf{X}\right) \\ &= \operatorname{Var}\left(\omega_{iA}^C \mid \mathbf{X}\right) + \operatorname{Var}\left(\omega_{iB}^C \mid \mathbf{X}\right) - 2\operatorname{Cov}\left(\omega_{iA}^C, \omega_{iB}^C \mid \mathbf{X}\right) \\ &= 2\operatorname{Var}\left(\omega_{ip}^C \mid \mathbf{X}\right) - 2\operatorname{Cov}\left(\omega_{iA}^C, \omega_{iB}^C \mid \mathbf{X}\right) \\ &= 2\left(\sigma^2_{\omega^C} - \psi^{CX}\right)\end{aligned} \tag{A50}$$

where $\sigma^2_{\omega^C}$ is the variance of the error term of the collapsed model, and ψ^{CX} is the average covariance between error terms for the same unit in the collapsed model. Substituting Equation (A50) into

3. here are no cross-sectional error correlations due to Assumption E.2, because randomization obviates the need to account for this kind of correlation.

Equation (A48) gives the variance of $\hat{\tau}$ in terms of parameters of the collapsed data:

$$\operatorname{Var}(\hat{\tau} \mid \mathbf{X}) = \left(\frac{2}{P(1-P)J}\right)\left(\sigma^2_{\omega^C} - \psi^{CX}\right) \tag{A51}$$

This formula is equal to the variance of $\hat{\tau}$ from the SCR formula applied to collapsed data, where $m = r = 1$. Applying the FP formula to collapsed data, however, gives the incorrect variance:

$$\operatorname{Var}(\hat{\tau} \mid \mathbf{X}) = \left(\frac{2}{P(1-P)J}\right)\sigma^2_{\omega^C} \tag{A52}$$

Equations (A51) and (A52) differ by the ψ^{CX} term that characterizes the covariance between the pre- and post-treatment error terms in the collapsed data. This term is omitted from the FP formula that (incorrectly) assumes no serial correlation in the error structure.

The error structure parameters of the collapsed data in Equation (A51) can also be expressed as functions of the parameters that define the error structure of the original, uncollapsed data

$$\begin{aligned}
\sigma^2_{\omega^C} &= \frac{1}{2}\operatorname{Var}\left(\omega^C_{iA} \mid \mathbf{X}\right) + \frac{1}{2}\operatorname{Var}\left(\omega^C_{iB} \mid \mathbf{X}\right) \\
&= \frac{1}{2}\operatorname{Var}\left(\frac{1}{r}\sum_{t=1}^{r}\omega_{it} \mid \mathbf{X}\right) + \frac{1}{2}\operatorname{Var}\left(\frac{1}{m}\sum_{s=-m+1}^{0}\omega_{is} \mid \mathbf{X}\right) \\
&= \frac{1}{2r}\left[\sigma^2_\omega + (r-1)\psi^A\right] + \frac{1}{2m}\left[\sigma^2_\omega + (m-1)\psi^B\right] \\
&= \frac{1}{2}\left[\left(\frac{m+r}{mr}\right)\sigma^2_\omega + \left(\frac{r-1}{r}\right)\psi^A + \left(\frac{m-1}{m}\right)\psi^B\right]
\end{aligned} \tag{A53}$$

$$\begin{aligned}
\psi^{CX} &= \frac{1}{J}\sum_{i=1}^{J}\operatorname{Cov}\left(\omega^C_{iA}, \omega^C_{iB} \mid \mathbf{X}\right) \\
&= \frac{1}{J}\sum_{i=1}^{J}\operatorname{Cov}\left(\frac{1}{r}\sum_{t=1}^{r}\omega_{it}, \frac{1}{m}\sum_{s=-m+1}^{0}\omega_{is} \mid \mathbf{X}\right) \\
&= \frac{1}{Jmr}\sum_{i=1}^{J}\sum_{t=1}^{r}\sum_{s=-m+1}^{0}\operatorname{Cov}(\omega_{it}, \omega_{is} \mid \mathbf{X}) \\
&= \psi^X
\end{aligned} \tag{A54}$$

Substituting Equations (A53) and (A54) into Equation (A51) gives

$$\operatorname{Var}(\hat{\tau} \mid \mathbf{X}) = \left(\frac{1}{P(1-P)J}\right)\left[\left(\frac{m+r}{mr}\right)\sigma^2_\omega + \left(\frac{m-1}{m}\right)\psi^B + \left(\frac{r-1}{r}\right)\psi^A - 2\psi^X\right]$$

which is equivalent to the variance of $\hat{\tau}$ given in Equation (A31). Hence, the uncollapsed, collapsed, and first-difference models yield (virtually) equivalent MDEs when using the appropriate power calculation formula.[4] By contrast, applying the FP formula ignores the across-period serial correlation that remains after collapsing serially correlated data.

A.3 Arbitrary cross-sectional correlations

In this section, we provide proofs of Lemma 1 and Lemma 2.

Lemma 1. *In a cross-sectional model with random assignment to treatment,* $\frac{\sigma_\varepsilon^2}{P(1-P)J}$ *is an unbiased estimator of* $\text{Var}(\widehat{\tau} \mid \mathbf{X})$ *even if* $\text{E}[\varepsilon_i \varepsilon_j \mid \mathbf{X}] \neq 0$ *for some* $i \neq j$.

Proof We wish to demonstrate that the OLS variance estimator, which assumes that errors are independent across units, is an unbiased estimator of the true variance under non-i.i.d. errors when units are randomly assigned to treatment. To do this, consider the following setup:

There are J units, indexed by i. P fraction of these units is randomized into treatment, and $(1-P)$ fraction remain in the control group, such that $i \in [i, PJ]$ are treated, and $i \in [PJ+1, J]$ are control. We observe each unit only once. This yields a DGP of:

$$Y_i = \beta + \tau D_i + \omega_i$$

where Y_i is the outcome of interest for unit i, β is the expected outcome for control observations, τ is a treatment effect that is constant across all units, and ω_i is an idiosyncratic error term distributed $\mathcal{N}(0, \sigma_\omega^2)$. These errors need not be independently drawn.

Coefficient estimate The coefficient estimates from an OLS regression are

$$\begin{aligned}
\begin{pmatrix} \widehat{\beta} \\ \widehat{\tau} \end{pmatrix} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'Y \\
&= \begin{pmatrix} J & PJ \\ PJ & PJ \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^J Y_i \\ \sum_{i=1}^J D_i Y_i \end{pmatrix} \\
&= \frac{1}{P(1-P)J^2} \begin{pmatrix} PJ\left(\sum_{i=1}^J Y_i - \sum_{i=1}^J D_i Y_i\right) \\ J\left(\sum_{i=1}^J D_i Y_i - P\sum_{i=1}^J Y_i\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{(1-P)J}\sum_{i=1}^J (1-D_i)Y_i \\ \frac{1}{PJ}\sum_{i=1}^J D_i Y_i - \frac{1}{(1-P)J}\sum_{i=1}^J (1-D_i)Y_i \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{(1-P)J}\sum_{i=PJ+1}^J Y_i \\ \frac{1}{PJ}\sum_{i=1}^{PJ} Y_i - \frac{1}{(1-P)J}\sum_{i=PJ+1}^J Y_i \end{pmatrix}
\end{aligned}$$

4. The only difference between these three MDE calculations is the critical values $t_{1-\kappa}^d$ and $t_{\alpha/2}^d$. The uncollapsed model will apply the CRVE *ex post*, which implies $d = J$ degrees of freedom *ex ante*. The collapsed model will apply the OLS variance estimator *ex post* to a panel with $2J$ observations and $J+2$ regressors, which implies $d = J-2$ degrees of freedom *ex ante*. The first-difference model will apply the OLS variance estimator *ex post* to a cross-sectional specification with J observations and 2 regressors, which implies $d = J-2$ degrees of freedom *ex ante*.

Defining the mean outcome in the treatment and control groups, respectively, as

$$\overline{Y}_T = \frac{1}{PJ}\sum_{i=1}^{PJ} Y_i$$
$$\overline{Y}_C = \frac{1}{(1-P)J}\sum_{i=PJ+1}^{J} Y_i$$

gives coefficient estimates of

$$\widehat{\beta} = \overline{Y}_C$$
$$\widehat{\tau} = \overline{Y}_T - \overline{Y}_C$$

Variance of coefficient estimate Note first that the OLS variance estimator is:

$$\widehat{\mathrm{Var}}_{OLS}\left(\widehat{\tau} \mid \mathbf{X}\right) = \frac{\sigma_\omega^2}{P(1-P)J}$$

The variance of the estimate of the treatment effect, $\hat{\tau}$, is:

$$\mathrm{Var}\left(\widehat{\tau} \mid \mathbf{X}\right) = \mathrm{Var}\left(\overline{Y}_T \mid \mathbf{X}\right) + \mathrm{Var}\left(\overline{Y}_C \mid \mathbf{X}\right) - 2\,\mathrm{Cov}\left(\overline{Y}_T, \overline{Y}_C \mid \mathbf{X}\right) \tag{A55}$$

where the first term of Equation (A55) is:

$$\begin{aligned}
\mathrm{Var}\left(\overline{Y}_T \mid \mathbf{X}\right) &= \mathrm{Var}\left(\frac{1}{PJ}\sum_{i=1}^{PJ} Y_i \mid \mathbf{X}\right) \\
&= \frac{1}{(PJ)^2}\mathrm{Var}\left(\sum_{i=1}^{PJ} Y_i \mid \mathbf{X}\right) \\
&= \frac{1}{(PJ)^2}\sum_{i=1}^{PJ}\mathrm{Var}(Y_i \mid \mathbf{X}) + \frac{2}{(PJ)^2}\sum_{i=1}^{PJ-1}\sum_{j=i+1}^{PJ}\mathrm{Cov}(Y_i, Y_j \mid \mathbf{X}) \\
&= \frac{1}{(PJ)^2}\sum_{i=1}^{PJ}\mathrm{Var}(\omega_i \mid \mathbf{X}) + \frac{2}{(PJ)^2}\sum_{i=1}^{PJ-1}\sum_{j=i+1}^{PJ}\mathrm{Cov}(\omega_i, \omega_j \mid \mathbf{X}) \\
&= \frac{\sigma_\omega^2}{PJ} + \frac{PJ-1}{PJ}\psi_T
\end{aligned} \tag{A56}$$

where

$$\psi_T \equiv \frac{2}{PJ(PJ-1)}\sum_{i=1}^{PJ-1}\sum_{j=i+1}^{PJ}\mathrm{Cov}\left(\omega_i, \omega_j \mid \mathbf{X}\right)$$

is the average covariance between treated units. The second term of Equation (A55) is:

$$\mathrm{Var}\left(\overline{Y}_C \mid \mathbf{X}\right) = \frac{\sigma_\omega^2}{(1-P)J} + \frac{(1-P)J-1}{(1-P)J}\psi_C \tag{A57}$$

where

$$\psi_C \equiv \frac{2}{(1-P)J((1-P)J-1)} \sum_{i=PJ+1}^{J-1} \sum_{j=i+1}^{J} \operatorname{Cov}(\omega_i, \omega_j \mid \mathbf{X})$$

is the average covariance between control units. The third term of Equation (A55) is:

$$\begin{aligned}
-2\operatorname{Cov}\left(\overline{Y}_T, \overline{Y}_C \mid \mathbf{X}\right) &= -2\operatorname{Cov}\left(\frac{1}{PJ}\sum_{i=1}^{PJ} Y_i, \frac{1}{(1-P)J}\sum_{j=PJ+1}^{J} Y_j \mid \mathbf{X}\right) \\
&= -2\sum_{i=1}^{PJ}\sum_{j=PJ+1}^{J} \operatorname{Cov}\left(\frac{1}{PJ}Y_i, \frac{1}{(1-P)J}Y_j \mid \mathbf{X}\right) \\
&= \frac{-2}{P(1-P)J^2}\sum_{i=1}^{PJ}\sum_{j=PJ+1}^{J} \operatorname{Cov}(\omega_i, \omega_j \mid \mathbf{X}) \\
&= -2\psi_{TC} \qquad \text{(A58)}
\end{aligned}$$

where

$$\psi_{TC} \equiv \frac{1}{P(1-P)J^2}\sum_{i=1}^{PJ}\sum_{j=PJ+1}^{J} \operatorname{Cov}(\omega_i, \omega_j \mid \mathbf{X})$$

is the average covariance between treatment and control units. Substituting Equations (A56)–(A58) into Equation (A55) yields:

$$\begin{aligned}
\operatorname{Var}(\widehat{\tau} \mid \mathbf{X}) &= \left[\frac{\sigma_\omega^2}{PJ} + \frac{PJ-1}{PJ}\psi_T\right] + \left[\frac{\sigma_\omega^2}{(1-P)J} + \frac{(1-P)J-1}{(1-P)J}\psi_C\right] - 2\psi_{TC} \\
&= \frac{\sigma_\omega^2}{P(1-P)J} + \frac{PJ-1}{PJ}\psi_T + \frac{(1-P)J-1}{(1-P)J}\psi_C - 2\psi_{TC}
\end{aligned}$$

Note that σ_ω^2, P, and J are constant population or design parameters. With this in mind, taking expectations yields:

$$\begin{aligned}
\operatorname{E}[\operatorname{Var}(\widehat{\tau} \mid \mathbf{X})] &= \frac{\sigma_\omega^2}{P(1-P)J} + \frac{PJ-1}{PJ}\operatorname{E}[\psi_T \mid \mathbf{X}] \\
&\quad + \frac{(1-P)J-1}{(1-P)J}\operatorname{E}[\psi_C \mid \mathbf{X}] - 2\operatorname{E}[\psi_{TC} \mid \mathbf{X}] \qquad \text{(A59)}
\end{aligned}$$

where, by random assignment to treatment

$$\begin{aligned}
\operatorname{E}[\psi_T \mid \mathbf{X}] &= \frac{2}{PJ(PJ-1)}\operatorname{E}\left[\sum_{i=1}^{PJ-1}\sum_{j=i+1}^{PJ} \operatorname{Cov}(\omega_i, \omega_j \mid \mathbf{X})\right] \\
&= \frac{2}{PJ(PJ-1)}\operatorname{E}\left[\sum_{i=1}^{J-1}\sum_{j=i+1}^{J} \operatorname{Cov}(\omega_i, \omega_j \mid \mathbf{X})\,\mathbf{1}\{i \in T, j \in T\}\right]
\end{aligned}$$

$$= \frac{2}{PJ(PJ-1)} \sum_{i=1}^{J-1} \sum_{j=i+1}^{J} \mathrm{Cov}(\omega_i, \omega_j \mid \mathbf{X}) \, \mathrm{E}[\mathbf{1}\{i \in T, j \in T\}]$$

$$= \frac{2}{PJ(PJ-1)} \sum_{i=1}^{J-1} \sum_{j=i+1}^{J} \mathrm{Cov}(\omega_i, \omega_j \mid \mathbf{X}) \, P^2$$

$$= \frac{2P}{J(PJ-1)} \sum_{i=1}^{J-1} \sum_{j=i+1}^{J} \mathrm{Cov}(\omega_i, \omega_j \mid \mathbf{X}) \tag{A60}$$

$$\mathrm{E}[\psi_C \mid \mathbf{X}] = \frac{2(1-P)}{J((1-P)J-1)} \sum_{i=1}^{J-1} \sum_{j=i+1}^{J} \mathrm{Cov}(\omega_i, \omega_j \mid \mathbf{X}) \tag{A61}$$

$$\mathrm{E}[\psi_{TC} \mid \mathbf{X}] = \frac{2}{J^2} \sum_{i=1}^{J-1} \sum_{j=i+1}^{J} \mathrm{Cov}(\omega_i, \omega_j \mid \mathbf{X}) \tag{A62}$$

Substituting Equations (A60)–(A62) into Equation (A59) yields:

$$\begin{aligned} \mathrm{E}[\mathrm{Var}(\widehat{\tau} \mid \mathbf{X})] &= \frac{\sigma_\omega^2}{P(1-P)J} + \frac{2}{J^2} \sum_{i=1}^{J-1} \sum_{j=i+1}^{J} \mathrm{Cov}(\omega_i, \omega_j \mid \mathbf{X}) \\ &\quad + \frac{2}{J^2} \sum_{i=1}^{J-1} \sum_{j=i+1}^{J} \mathrm{Cov}(\omega_i, \omega_j \mid \mathbf{X}) - \frac{4}{J^2} \sum_{i=1}^{J-1} \sum_{j=i+1}^{J} \mathrm{Cov}(\omega_i, \omega_j \mid \mathbf{X}) \\ &= \frac{\sigma_\omega^2}{P(1-P)J} \\ &= \widehat{\mathrm{Var}}_{OLS}(\widehat{\tau} \mid \mathbf{X}) \end{aligned}$$

Therefore, the OLS variance estimator is an unbiased estimate of the true variance under random assignment to treatment, even under non-i.i.d. errors. □

Lemma 2. *In a panel difference-in-differences model with treatment randomly assigned at the unit level,* $\left(\frac{1}{P(1-P)J}\right) \left[\left(\frac{m+r}{mr}\right) \sigma_\omega^2 + \left(\frac{m-1}{m}\right) \psi^B + \left(\frac{r-1}{r}\right) \psi^A - 2\psi^X \right]$ *is an unbiased estimator of* $\mathrm{Var}(\hat{\tau} \mid \mathbf{X})$*, even in the presence of arbitrary within-period cross-sectional correlations.*

Proof We wish to demonstrate that the serial-correlation-robust variance, which assumes that errors are independent across units, is an unbiased estimator of the true variance under arbitrary within-period correlations, when units are randomly assigned to treatment. To do this, consider the following setup:

There are J units, indexed by i. P fraction of these units is randomized into treatment, and $(1-P)$ fraction remain in the control group, such that $i \in [i, PJ]$ are treated, and $i \in [PJ+1, J]$ are control. We observe each unit m times in the pre-treatment period, and r times in the post-treatment period, such that time periods $t \in [-m+1, 0]$ are in the pre-treatment period, and time periods $s \in [1, r]$ are in the post-treatment period. Treatment begins at the same time for all treated units. There are unit-specific shocks that affect all time periods, as well as time-varying shocks that are common across units. Treatment begins at the same time for all treated units. This yields a DGP of:

$$\begin{aligned} Y_{it} &= \beta + \tau D_{it} + \varepsilon_{it} \\ &= \beta + \tau D_{it} + \upsilon_i + \delta_t + \omega_{it} \end{aligned}$$

where Y_{it} is the outcome of interest for unit i at time t; β is the expected outcome of non-treated observations; τ is a treatment effect that is constant across all units and time periods; D_{it} is a time-varying treatment indicator, equal to one if and only if a unit is in the treatment group and $t \geq 1$; υ_i is a time-invariant unit-specific effect distributed i.i.d. $\mathcal{N}(0, \sigma_\upsilon^2)$; δ_t is a time-specific effect distributed i.i.d. $\mathcal{N}(0, \sigma_\delta^2)$; and ω_{it} is an idiosyncratic error term distributed $\mathcal{N}(0, \sigma_\omega^2)$. These errors need not be independently drawn, such that ω_{it} may exhibit arbitrary within-unit and within-period correlations. However, this still assumes $\mathrm{E}\left[\omega_{it}, \omega_{jt} \mid \mathbf{X}\right] = 0$ in all cases where $i \neq j$ and $t \neq s$.

Coefficient estimate Under this model, the estimate of the treatment effect is

$$\widehat{\tau} = \left(\overline{Y}_T^A - \overline{Y}_T^B\right) - \left(\overline{Y}_C^A - \overline{Y}_C^B\right)$$

where

$$\begin{aligned} \overline{Y}_T^A &= \frac{1}{PJr} \sum_{i=1}^{PJ} \sum_{t=1}^{r} Y_{it} \\ \overline{Y}_T^B &= \frac{1}{PJm} \sum_{i=1}^{PJ} \sum_{t=-m+1}^{0} Y_{it} \\ \overline{Y}_C^A &= \frac{1}{(1-P)Jr} \sum_{i=PJ+1}^{J} \sum_{t=1}^{r} Y_{it} \\ \overline{Y}_C^B &= \frac{1}{(1-P)Jm} \sum_{i=PJ+1}^{J} \sum_{t=-m+1}^{0} Y_{it} \end{aligned}$$

Variance of coefficient estimate Note first that the serial-correlation-robust variance estimator is

$$\widehat{\mathrm{Var}}_{SCR}\left(\widehat{\tau} \mid \mathbf{X}\right) = \left(\frac{1}{P(1-P)J}\right) \left[\left(\frac{m+r}{mr}\right) \sigma_\omega^2 + \left(\frac{m-1}{m}\right) \psi^B + \left(\frac{r-1}{r}\right) \psi^A - 2\psi^X\right]$$

We wish to demonstrate that this is an unbiased estimator of the variance of $\hat{\tau}$ when units have been randomly assigned to treatment but may exhibit between-unit error correlations with a given time period.

The variance of the treatment effect estimator is:

$$\mathrm{Var}\left(\widehat{\tau} \mid \mathbf{X}\right) = \mathrm{Var}\left(\overline{Y}_T^A \mid \mathbf{X}\right) + \mathrm{Var}\left(\overline{Y}_T^B \mid \mathbf{X}\right) + \mathrm{Var}\left(\overline{Y}_C^A \mid \mathbf{X}\right) + \mathrm{Var}\left(\overline{Y}_C^B \mid \mathbf{X}\right) - 2\Big[\mathrm{Cov}\left(\overline{Y}_T^A, \overline{Y}_T^B \mid \mathbf{X}\right)$$

$$+ \operatorname{Cov}\left(\overline{Y}_T^A, \overline{Y}_C^A \mid \mathbf{X}\right) + \operatorname{Cov}\left(\overline{Y}_T^B, \overline{Y}_C^B \mid \mathbf{X}\right) + \operatorname{Cov}\left(\overline{Y}_C^A, \overline{Y}_C^B \mid \mathbf{X}\right) \Bigg] \tag{A63}$$

This equation can be broken up into components, such that:

$$\begin{aligned}
\operatorname{Var}\left(\overline{Y}_T^A \mid \mathbf{X}\right) &= \operatorname{Var}\left(\frac{1}{PJr}\sum_{i=1}^{PJ}\sum_{t=1}^{r} Y_{it} \mid \mathbf{X}\right) \\
&= \frac{1}{(PJr)^2}\operatorname{Var}\left(\sum_{i=1}^{PJ}\sum_{t=1}^{r} Y_{it} \mid \mathbf{X}\right) \\
&= \frac{1}{(PJr)^2}\Bigg[PJr\operatorname{Var}(Y_{it} \mid \mathbf{X}) + 2\sum_{i=1}^{PJ}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\operatorname{Cov}(Y_{it}, Y_{is} \mid \mathbf{X}) \qquad \text{(A64)} \\
&\quad + 2\sum_{t=1}^{r}\sum_{i=1}^{PJ-1}\sum_{j=i+1}^{PJ}\operatorname{Cov}(Y_{it}, Y_{jt} \mid \mathbf{X})\Bigg] \\
&= \frac{\sigma_\varepsilon^2}{PJr} + \frac{2}{(PJr)^2}\sum_{i=1}^{PJ}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\left(\sigma_\upsilon^2 + \operatorname{Cov}(\omega_{it}, \omega_{is} \mid \mathbf{X})\right) \\
&\quad + \frac{2}{(PJr)^2}\sum_{t=1}^{r}\sum_{i=1}^{PJ-1}\sum_{j=i+1}^{PJ}\left(\sigma_\delta^2 + \operatorname{Cov}(\omega_{it}, \omega_{jt} \mid \mathbf{X})\right) \\
&= \frac{1}{PJr}\left(r\sigma_\upsilon^2 + PJ\sigma_\delta^2 + \sigma_\omega^2\right) + \frac{2}{(PJr)^2}\sum_{i=1}^{PJ}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\operatorname{Cov}(\omega_{it}, \omega_{is} \mid \mathbf{X}) \\
&\quad + \frac{2}{(PJr)^2}\sum_{t=1}^{r}\sum_{i=1}^{PJ-1}\sum_{j=i+1}^{PJ}\operatorname{Cov}(\omega_{it}, \omega_{jt} \mid \mathbf{X}) \\
&= \frac{1}{PJr}\left(r\sigma_\upsilon^2 + PJ\sigma_\delta^2 + \sigma_\omega^2 + (r-1)\psi_{i,T}^A + (PJ-1)\psi_{t,T}^A\right) \qquad \text{(A65)}
\end{aligned}$$

where

$$\psi_{i,T}^A \equiv \frac{2}{PJr(r-1)}\sum_{i=1}^{PJ}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\operatorname{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right)$$

$$\psi_{t,T}^A \equiv \frac{2}{PJ(PJ-1)r}\sum_{i=1}^{PJ-1}\sum_{j=i+1}^{PJ}\sum_{t=1}^{r}\operatorname{Cov}\left(\omega_{it}, \omega_{jt} \mid \mathbf{X}\right)$$

Similarly, with arbitrary correlations, the remaining variance terms of Equation (A63) are

$$\operatorname{Var}\left(\overline{Y}_T^B \mid \mathbf{X}\right) = \frac{1}{PJm}\left(m\sigma_\upsilon^2 + PJ\sigma_\delta^2 + \sigma_\omega^2 + (m-1)\psi_{i,T}^B + (PJ-1)\psi_{t,T}^B\right) \tag{A66}$$

where

$$\psi_{i,T}^B \equiv \frac{2}{PJm(m-1)}\sum_{i=1}^{PJ}\sum_{t=-m+1}^{-1}\sum_{s=t+1}^{0}\operatorname{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right)$$

$$\psi^B_{t,T} \equiv \frac{2}{PJ(PJ-1)m} \sum_{i=1}^{PJ-1} \sum_{j=i+1}^{PJ} \sum_{t=-m+1}^{0} \text{Cov}\left(\omega_{it}, \omega_{jt} \mid \mathbf{X}\right)$$

$$\text{Var}\left(\overline{Y}^A_C \mid \mathbf{X}\right) = \frac{1}{(1-P)Jr}\left(r\sigma^2_v + (1-P)J\sigma^2_\delta + \sigma^2_\omega + (r-1)\psi^A_{i,C} + ((1-P)J-1)\psi^A_{t,C}\right) \tag{A67}$$

where

$$\psi^A_{i,C} \equiv \frac{2}{(1-P)Jr(r-1)} \sum_{i=PJ+1}^{J} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \text{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right)$$
$$\psi^A_{t,C} \equiv \frac{2}{(1-P)J((1-P)J-1)r} \sum_{i=PJ+1}^{J-1} \sum_{j=i+1}^{J} \sum_{t=1}^{r} \text{Cov}\left(\omega_{it}, \omega_{jt} \mid \mathbf{X}\right)$$

$$\text{Var}\left(\overline{Y}^B_C \mid \mathbf{X}\right) = \frac{1}{(1-P)Jm}\left(m\sigma^2_v + (1-P)J\sigma^2_\delta + \sigma^2_\omega + (m-1)\psi^B_{i,C} + ((1-P)J-1)\psi^B_{t,C}\right) \tag{A68}$$

where

$$\psi^B_{i,C} \equiv \frac{2}{(1-P)Jm(m-1)} \sum_{i=PJ+1}^{J} \sum_{t=-m+1}^{-1} \sum_{s=t+1}^{0} \text{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right)$$
$$\psi^B_{t,C} \equiv \frac{2}{(1-P)J((1-P)J-1)m} \sum_{i=PJ+1}^{J-1} \sum_{j=i+1}^{J} \sum_{t=-m+1}^{0} \text{Cov}\left(\omega_{it}, \omega_{jt} \mid \mathbf{X}\right) \tag{A69}$$

With arbitrary correlations, the first covariance component of Equation (A63) is

$$\begin{aligned}
\text{Cov}\left(\overline{Y}^A_T, \overline{Y}^B_T \mid \mathbf{X}\right) &= \text{E}\left[\overline{Y}^A_T \overline{Y}^B_T \mid \mathbf{X}\right] - \text{E}\left[\overline{Y}^A_T \mid \mathbf{X}\right] \text{E}\left[\overline{Y}^B_T \mid \mathbf{X}\right] \\
&= \left[\beta(\beta+\tau) + \frac{1}{PJ}\text{E}\left[v^2_i \mid \mathbf{X}\right] + \frac{1}{(PJ)^2 mr} \sum_{i=1}^{PJ} \sum_{t=-m+1}^{0} \sum_{s=1}^{r} \text{E}[\omega_{it}\omega_{is} \mid \mathbf{X}]\right] - \beta(\beta+\tau) \\
&= \frac{\sigma^2_v}{PJ} + \frac{1}{(PJ)^2 mr} \sum_{i=1}^{PJ} \sum_{t=-m+1}^{0} \sum_{s=1}^{r} \text{Cov}(\omega_{it}, \omega_{is} \mid \mathbf{X}) \\
&= \frac{1}{PJ}\left(\sigma^2_v + \psi^X_{i,T}\right)
\end{aligned} \tag{A70}$$

where

$$\psi^X_{i,T} \equiv \frac{1}{PJmr} \sum_{i=1}^{PJ} \sum_{t=-m+1}^{0} \sum_{s=1}^{r} \text{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right) \tag{A71}$$

Similarly, with arbitrary correlations, the remaining covariance terms of Equation (A63) are

$$\mathrm{Cov}\left(\overline{Y}_T^A, \overline{Y}_C^A \mid \mathbf{X}\right) = \frac{1}{r}\left(\sigma_\delta^2 + \psi_{t,TC}^A\right) \tag{A72}$$

where

$$\psi_{t,TC}^A \equiv \frac{1}{P(1-P)J^2 r} \sum_{i=1}^{PJ} \sum_{j=PJ+1}^{J} \sum_{t=1}^{r} \mathrm{Cov}\left(\omega_{it}, \omega_{jt} \mid \mathbf{X}\right)$$

$$\mathrm{Cov}\left(\overline{Y}_T^B, \overline{Y}_C^B \mid \mathbf{X}\right) = \frac{1}{m}\left(\sigma_\delta^2 + \psi_{t,TC}^B\right) \tag{A73}$$

where

$$\psi_{t,TC}^B \equiv \frac{1}{P(1-P)J^2 m} \sum_{i=1}^{PJ} \sum_{j=PJ+1}^{J} \sum_{t=-m+1}^{0} \mathrm{Cov}\left(\omega_{it}, \omega_{jt} \mid \mathbf{X}\right)$$

$$\mathrm{Cov}\left(\overline{Y}_C^A, \overline{Y}_C^B \mid \mathbf{X}\right) = \frac{1}{(1-P)J}\left(\sigma_v^2 + \psi_{i,C}^X\right) \tag{A74}$$

where

$$\psi_{i,C}^X \equiv \frac{1}{(1-P)Jmr} \sum_{i=PJ+1}^{J} \sum_{t=-m+1}^{0} \sum_{s=1}^{r} \mathrm{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right) \tag{A75}$$

Substituting Equations (A65)–(A74) into Equation (A63) and simplifying gives

$$\begin{aligned}
\mathrm{Var}\left(\widehat{\tau} \mid \mathbf{X}\right) = & \frac{1}{PJr}\left(r\sigma_v^2 + PJ\sigma_\delta^2 + \sigma_\omega^2 + (r-1)\psi_{i,T}^A + (PJ-1)\psi_{t,T}^A\right) \\
& + \frac{1}{PJm}\left(m\sigma_v^2 + PJ\sigma_\delta^2 + \sigma_\omega^2 + (m-1)\psi_{i,T}^B + (PJ-1)\psi_{t,T}^B\right) \\
& + \frac{1}{(1-P)Jr}\left(r\sigma_v^2 + (1-P)J\sigma_\delta^2 + \sigma_\omega^2 + (r-1)\psi_{i,C}^A + ((1-P)J-1)\psi_{t,C}^A\right) \\
& + \frac{1}{(1-P)Jm}\left(m\sigma_v^2 + (1-P)J\sigma_\delta^2 + \sigma_\omega^2 + (m-1)\psi_{i,C}^B + ((1-P)J-1)\psi_{t,C}^B\right) \\
& - 2\left[\frac{1}{PJ}\left(\sigma_v^2 + \psi_{i,T}^X\right) + \frac{1}{r}\left(\sigma_\delta^2 + \psi_{t,X}^A\right) + \frac{1}{m}\left(\sigma_\delta^2 + \psi_{t,X}^B\right) + \frac{1}{(1-P)J}\left(\sigma_v^2 + \psi_{i,C}^X\right)\right] \\
= & \left(\frac{m+r}{P(1-P)Jmr}\right)\sigma_\omega^2 + \left(\frac{m-1}{PJm}\right)\psi_{i,T}^B + \left(\frac{PJ-1}{PJm}\right)\psi_{t,T}^B \\
& + \left(\frac{r-1}{PJr}\right)\psi_{i,T}^A + \left(\frac{PJ-1}{PJr}\right)\psi_{t,T}^A + \left(\frac{m-1}{(1-P)Jm}\right)\psi_{i,C}^B \\
& + \left(\frac{(1-P)J-1}{(1-P)Jm}\right)\psi_{t,C}^B + \left(\frac{r-1}{(1-P)Jr}\right)\psi_{i,C}^A + \left(\frac{(1-P)J-1}{(1-P)Jr}\right)\psi_{t,C}^A
\end{aligned}$$

$$- \frac{2}{PJ}\psi^X_{i,T} - \frac{2}{m}\psi^B_{t,X} - \frac{2}{r}\psi^A_{t,X} - \frac{2}{(1-P)J}\psi^X_{i,C}$$

Next, we show that, in expectation, this is equal to $\widehat{\text{Var}}_{SCR}(\widehat{\tau} \mid \mathbf{X})$. We begin by taking expectations:

$$\begin{aligned}
\mathrm{E}\left[\mathrm{Var}\left(\widehat{\tau} \mid \mathbf{X}\right)\right] = & \left(\frac{m+r}{P(1-P)Jmr}\right)\sigma^2_\omega + \left(\frac{m-1}{PJm}\right)\mathrm{E}\left[\psi^B_{i,T} \mid \mathbf{X}\right] + \left(\frac{PJ-1}{PJm}\right)\mathrm{E}\left[\psi^B_{t,T} \mid \mathbf{X}\right] \\
& + \left(\frac{r-1}{PJr}\right)\mathrm{E}\left[\psi^A_{i,T} \mid \mathbf{X}\right] + \left(\frac{PJ-1}{PJr}\right)\mathrm{E}\left[\psi^A_{t,T} \mid \mathbf{X}\right] + \left(\frac{m-1}{(1-P)Jm}\right)\mathrm{E}\left[\psi^B_{i,C} \mid \mathbf{X}\right] \\
& + \left(\frac{(1-P)J-1}{(1-P)Jm}\right)\mathrm{E}\left[\psi^B_{t,C} \mid \mathbf{X}\right] + \left(\frac{r-1}{(1-P)Jr}\right)\mathrm{E}\left[\psi^A_{i,C} \mid \mathbf{X}\right] + \\
& + \left(\frac{(1-P)J-1}{(1-P)Jr}\right)\mathrm{E}\left[\psi^A_{t,C} \mid \mathbf{X}\right] - \frac{2}{PJ}\mathrm{E}\left[\psi^X_{i,T} \mid \mathbf{X}\right] - \frac{2}{m}\mathrm{E}\left[\psi^B_{t,X} \mid \mathbf{X}\right] \\
& - \frac{2}{r}\mathrm{E}\left[\psi^A_{t,X} \mid \mathbf{X}\right] - \frac{2}{(1-P)J}\mathrm{E}\left[\psi^X_{i,C} \mid \mathbf{X}\right]
\end{aligned}$$

where

$$\mathrm{E}\left[\psi^B_{i,T} \mid \mathbf{X}\right] = \frac{2}{Jm(m-1)}\sum_{i=1}^{J}\sum_{t=-m+1}^{-1}\sum_{s=t+1}^{0}\mathrm{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right)$$

$$\mathrm{E}\left[\psi^B_{t,T} \mid \mathbf{X}\right] = \frac{2P}{J(PJ-1)m}\sum_{i=1}^{J-1}\sum_{j=i+1}^{J}\sum_{t=-m+1}^{0}\mathrm{Cov}\left(\omega_{it}, \omega_{jt} \mid \mathbf{X}\right)$$

$$\mathrm{E}\left[\psi^A_{i,T} \mid \mathbf{X}\right] = \frac{2}{Jr(r-1)}\sum_{i=1}^{J}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right)$$

$$\mathrm{E}\left[\psi^A_{t,T} \mid \mathbf{X}\right] = \frac{2P}{J(PJ-1)r}\sum_{i=1}^{J-1}\sum_{j=i+1}^{J}\sum_{t=1}^{r}\mathrm{Cov}\left(\omega_{it}, \omega_{jt} \mid \mathbf{X}\right)$$

$$\mathrm{E}\left[\psi^B_{i,C} \mid \mathbf{X}\right] = \frac{2}{Jm(m-1)}\sum_{i=1}^{J}\sum_{t=-m+1}^{-1}\sum_{s=t+1}^{0}\mathrm{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right)$$

$$\mathrm{E}\left[\psi^B_{t,C} \mid \mathbf{X}\right] = \frac{2(1-P)}{J((1-P)J-1)m}\sum_{i=1}^{J-1}\sum_{j=i+1}^{J}\sum_{t=-m+1}^{0}\mathrm{Cov}\left(\omega_{it}, \omega_{jt} \mid \mathbf{X}\right)$$

$$\mathrm{E}\left[\psi^A_{i,C} \mid \mathbf{X}\right] = \frac{2}{Jr(r-1)}\sum_{i=1}^{J}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right)$$

$$\mathrm{E}\left[\psi^A_{t,C} \mid \mathbf{X}\right] = \frac{2(1-P)}{J((1-P)J-1)r}\sum_{i=1}^{J-1}\sum_{j=i+1}^{J}\sum_{t=1}^{r}\mathrm{Cov}\left(\omega_{it}, \omega_{jt} \mid \mathbf{X}\right)$$

$$\mathrm{E}\left[\psi^X_{i,T} \mid \mathbf{X}\right] = \frac{1}{Jmr}\sum_{i=1}^{J}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right)$$

$$\mathrm{E}\left[\psi^B_{t,TC} \mid \mathbf{X}\right] = \frac{2}{J^2m}\sum_{i=1}^{J-1}\sum_{j=i+1}^{J}\sum_{t=-m+1}^{0}\mathrm{Cov}\left(\omega_{it}, \omega_{jt} \mid \mathbf{X}\right)$$

$$\mathrm{E}\left[\psi^A_{t,TC} \mid \mathbf{X}\right] = \frac{2}{J^2 r}\sum_{i=1}^{J-1}\sum_{j=i+1}^{J}\sum_{t=1}^{r}\mathrm{Cov}\left(\omega_{it},\omega_{jt}\mid \mathbf{X}\right)$$

$$\mathrm{E}\left[\psi^X_{i,C} \mid \mathbf{X}\right] = \frac{1}{Jmr}\sum_{i=1}^{J}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}\left(\omega_{it},\omega_{is}\mid \mathbf{X}\right)$$

Furthermore, define

$$\psi^B \equiv \frac{2}{Jm(m-1)}\sum_{i=1}^{J}\sum_{t=-m+1}^{-1}\sum_{s=t+1}^{0}\mathrm{Cov}\left(\omega_{it},\omega_{is}\mid \mathbf{X}\right)$$

$$\psi^A \equiv \frac{2}{Jr(r-1)}\sum_{i=1}^{J}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}\left(\omega_{it},\omega_{is}\mid \mathbf{X}\right)$$

$$\psi^X \equiv \frac{1}{Jmr}\sum_{i=1}^{J}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}\left(\omega_{it},\omega_{is}\mid \mathbf{X}\right)$$

Using these terms, we can rewrite:

$$\begin{aligned}
\mathrm{E}\left[\mathrm{Var}\left(\widehat{\tau}\mid \mathbf{X}\right)\right] = &\left(\frac{m+r}{P(1-P)Jmr}\right)\sigma^2_\omega + \left(\frac{m-1}{PJm}\right)\psi^B \\
&+ \left(\frac{pJ-1}{PJm}\right)\left(\frac{2P}{J(PJ-1)m}\right)\sum_{i=1}^{J-1}\sum_{j=i+1}^{J}\sum_{t=-m+1}^{0}\mathrm{Cov}\left(\omega_{it},\omega_{jt}\mid \mathbf{X}\right) \\
&+ \left(\frac{r-1}{PJr}\right)\psi^A + \left(\frac{PJ-1}{PJr}\right)\left(\frac{2P}{J(PJ-1)r}\right)\sum_{i=1}^{J-1}\sum_{j=i+1}^{J}\sum_{t=1}^{r}\mathrm{Cov}\left(\omega_{it},\omega_{jt}\mid \mathbf{X}\right) \\
&+ \left(\frac{m-1}{(1-P)Jm}\right)\psi^B \\
&+ \left(\frac{(1-P)J-1}{(1-P)Jm}\right)\left(\frac{2(1-P)}{J((1-P)J-1)m}\right)\sum_{i=1}^{J-1}\sum_{j=i+1}^{J}\sum_{t=-m+1}^{0}\mathrm{Cov}\left(\omega_{it},\omega_{jt}\mid \mathbf{X}\right) \\
&+ \left(\frac{r-1}{(1-P)Jr}\right)\psi^A \\
&+ \left(\frac{(1-P)J-1}{(1-P)Jr}\right)\left(\frac{2(1-P)}{J((1-P)J-1)r}\right)\sum_{i=1}^{J-1}\sum_{j=i+1}^{J}\sum_{t=1}^{r}\mathrm{Cov}\left(\omega_{it},\omega_{jt}\mid \mathbf{X}\right) \\
&- \left(\frac{2}{PJ}\right)\psi^X - \left(\frac{2}{m}\right)\left(\frac{2}{J^2 m}\right)\sum_{i=1}^{J-1}\sum_{j=i+1}^{J}\sum_{t=-m+1}^{0}\mathrm{Cov}\left(\omega_{it},\omega_{jt}\mid \mathbf{X}\right) \\
&- \left(\frac{2}{(1-P)J}\right)\psi^X - \left(\frac{2}{r}\right)\left(\frac{2}{J^2 r}\right)\sum_{i=1}^{J-1}\sum_{j=i+1}^{J}\sum_{t=1}^{r}\mathrm{Cov}\left(\omega_{it},\omega_{jt}\mid \mathbf{X}\right) \\
= &\left(\frac{m+r}{P(1-P)Jmr}\right)\sigma^2_\omega + \left[\frac{m-1}{PJm} + \frac{m-1}{(1-P)Jm}\right]\psi^B + \left[\frac{r-1}{PJr} + \frac{r-1}{(1-P)Jr}\right]\psi^A
\end{aligned}$$

$$
\begin{aligned}
&+ \left[\frac{-2}{PJ} - \frac{2}{(1-P)J}\right]\psi^X + \left[\frac{2}{J^2m^2} + \frac{2}{J^2m^2} - \frac{4}{J^2m^2}\right]\sum_{i=1}^{J-1}\sum_{j=i+1}^{J}\sum_{t=-m+1}^{0}\mathrm{Cov}\left(\omega_{it}, \omega_{jt} \mid \mathbf{X}\right) \\
&+ \left[\frac{2}{J^2r^2} + \frac{2}{J^2r^2} - \frac{4}{J^2r^2}\right]\sum_{i=1}^{J-1}\sum_{j=i+1}^{J}\sum_{t=1}^{r}\mathrm{Cov}\left(\omega_{it}{}', \omega_{jt} \mid \mathbf{X}\right) \\
= &\left(\frac{m+r}{P(1-P)Jmr}\right)\sigma^2_\omega + \left(\frac{m-1}{P(1-P)Jm}\right)\psi^B + \left(\frac{r-1}{P(1-P)Jr}\right)\psi^A - \frac{2}{P(1-P)J}\psi^X \\
= &\frac{1}{P(1-P)J}\left[\left(\frac{m+r}{mr}\right)\sigma^2_\omega + \left(\frac{m-1}{m}\right)\psi^B + \left(\frac{r-1}{r}\right)\psi^A - 2\psi^X\right] \\
= &\widehat{\mathrm{Var}}_{SCR}\left(\widehat{\tau} \mid \mathbf{X}\right)
\end{aligned}
$$

And therefore the SCR variance estimator is an unbiased estimate of the true variance under random assignment to treatment, even under non-i.i.d. errors. □

B Figures in main text

This section provides further detail on the simulations and power calculations referenced in the main text. We discuss the algorithms and assumptions behind each of the simulation plots, as well as the two analytical power calculation figures.

B.1 Simulated AR(1) data

In Figure 2, we run Monte Carlo simulations where each iteration generates a new simulated dataset with an idiosyncratic error term (ω_{it}) that evolves via an AR(1) process. We vary the following three parameters across sets of 10,000 simulations: the number of pre-treatment periods (m), the number of post-treatment periods (r), and the AR(1) dependece parameter (γ). The remaining parameters (J, P, α, κ, β, μ_υ, σ^2_υ, μ_δ, σ^2_δ, σ^2_ω) are fixed across all simulations.

Step 1: We calculate τ^{FP} and τ^{SCR} for each set of simulations, given a set of parameters values for m, r, and γ. These values are functions of the number of pre-treatment periods m, the number of post-treatment periods r, the number of units J, the proportion of units randomized into treatment P, the desired Type-I error rate α, the desired power κ, and the idiosyncratic variance σ^2_ω:

$$\tau^{FP} = \left(t^J_{1-\kappa} + t^J_{\alpha/2}\right)\sqrt{\left(\frac{\sigma^2_\omega}{P(1-P)J}\right)\left(\frac{m+r}{mr}\right)} \tag{B1}$$

$$\tau^{SCR} = (t^J_{1-\kappa} + t^J_{\alpha/2})\sqrt{\left(\frac{1}{P(1-P)J}\right)\left[\left(\frac{m+r}{mr}\right)\sigma^2_\omega + \left(\frac{m-1}{m}\right)\psi^B + \left(\frac{r-1}{r}\right)\psi^A - 2\psi^X\right]} \tag{B2}$$

Note that for both τ^{FP} and τ^{SCR}, we calculate the critical values $t^J_{1-\kappa}$ and $t^J_{\alpha/2}$ assuming J degrees of freedom, which is consistent with applying the CRVE *ex post*, clustering at the unit level with J units. Note also that ψ^B, ψ^A, and ψ^X depend on the correlation structure of the errors, and the AR(1) process enables us to derive closed form expressions for these covariances in terms of γ, σ^2_ω, m, and r. Because we set $m = r$ across all simulations, we can write ψ^B, ψ^A, and ψ^X as:

$$\psi^B = \frac{2\sigma^2_\omega}{(m-1)m}\sum_{z=1}^{m-1}(m-z)\gamma^z \tag{B3}$$

$$\psi^A = \psi^B \tag{B4}$$

$$\psi^X = \frac{\sigma^2_\omega}{m^2}\left[\sum_{z=1}^{m} z\gamma^z + \sum_{z=m+1}^{2m-1}(2m-z)\gamma^z\right] \tag{B5}$$

Step 2: For each simulation, we generate a dataset as specified by the data generating process:

$$Y_{it} = \beta + \upsilon_i + \delta_t + \omega_{it} \tag{B6}$$

To do this, we draw J independent values of υ_i from the distribution $N(\mu_\upsilon, \sigma^2_\upsilon)$, and draw $m + r$ independent values of δ_t from the distribution $N(\mu_\delta, \sigma^2_\delta)$. We create the idiosyncratic error $\omega_{it} = \gamma\omega_{i(t-1)} + \xi_{it}$ by simulating an AR(1) process with serial correlation γ and a white noise term ξ_{it}

drawn from the distribution $N(0, \sigma_\xi^2)$, where $\sigma_\xi^2 = \sigma_\omega^2(1-\gamma^2)$.[5]

Step 3: We randomly assign treatment to PJ units. This involves randomly scrambling a vector of PJ ones and $(1-P)J$ zeros and assigning each unit i either a 1 indicating treatment or a 0 indicating control.[6] This allows us to construct a time-varying treatment indicator D_{it}, where $D_{it} = 1$ for all treated units in post-treatment periods only, and $D_{it} = 0$ otherwise. We then create three outcome variables by adding treatment effects to the data generated in the previous step:

$$Y_{it}^{FP} \equiv Y_{it} + \tau^{FP} D_{it} \tag{B7}$$
$$Y_{it}^{SCR} \equiv Y_{it} + \tau^{SCR} D_{it} \tag{B8}$$
$$Y_{it}^{0} \equiv Y_{it} + \tau^{0} D_{it}, \tag{B9}$$

where $\tau^0 = 0$ is a placebo treatment effect.

Step 4: We estimate the following three OLS-fixed effects regressions:

$$Y_{it}^{FP} = \beta + \tau^{FP} D_{it} + \upsilon_i + \delta_t + \omega_{it} \tag{B10}$$
$$Y_{it}^{SCR} = \beta + \tau^{SCR} D_{it} + \upsilon_i + \delta_t + \omega_{it} \tag{B11}$$
$$Y_{it}^{0} = \beta + \tau^{0} D_{it} + \upsilon_i + \delta_t + \omega_{it} \tag{B12}$$

Step 5: For each estimated $\hat{\tau}^{FP}$, $\hat{\tau}^{SCR}$, and $\hat{\tau}^0$, we compute both OLS standard errors and CRVE standard errors, clustered at the unit level.

We repeat Steps 2–5 10,000 times, for values of $m = r \in \{1, \ldots, 20\}$ and for values of $\gamma \in \{0, 0.3, 0.5, 0.7, 0.9\}$.[7] After each set of 10,000 simulations, we calculate the percent of simulations where $\hat{\tau}^{FP}$, $\hat{\tau}^{SCR}$, and $\hat{\tau}^0$ reject null hypothesis of $\tau = 0$ at significance level α, under both OLS and CRVE standard errors.

Figure 2 reports these rejection rates on the vertical axes, with the number of pre- and post-treatment periods ($m = r$) on the horizontal axes, for each value of γ. Reading the top row left to right, we report the rejection rates for τ^{FP} under OLS standard errors, for τ^{FP} under CRVE standard errors, and for τ^{SCR} under CRVE standard errors, respectively. Because these are rejection rates of true effects, we interpret these curves as realized statistical power. Reading the bottom row left to right, we report the rejection rates for τ^0 under OLS standard errors, for τ^0 under CRVE standard errors, and for τ^0 under CRVE standard errors, respectively. Because these are rejection rates of placebo effects, we report these curves as realized false rejection rates. (The bottom-center and bottom-right panels report identical rejection rates, because the center and right columns have the same test for false rejection rates.)

We fix $\alpha = 0.05$ and $\kappa = 0.80$ across all simulations, as these are the critical values commonly used in practice. However, they are essentially arbitrary, and our simulation results would look identical if we had chosen alternative tolerances for Type I vs. Type II errors. (The only difference

5. We allow for a sufficiently long "burn-in period" in this AR(1) process, so that the process starts to evolve many periods before the first period of simulated data.

6. Our code rounds PJ to the nearest integer value, however PJ is already an integer in our main parameterization. Note that for the τ^{FP} and τ^{SCR} to be precisely calibrated, round$(PJ)/J$ needs to equal the parameter value P.

7. We set $m = r$ only for simplicity. However, the results are very similar if fix $m = 3$ and vary $r \in \{1, \ldots, 20\}$, or vice versa. In Appendix C, we present results that vary m and r separately.

would be that the vertical axes would change to reflect these alternative values.) All other fixed parameter values are arbitrary. We have set $J = 50$, $P = 0.5$, $\beta = 0$, $\mu_\upsilon = 100$, $\sigma^2_\upsilon = 80$, $\mu_\delta = 20$, $\sigma^2_\delta = 10$, and $\sigma^2_\omega = 10$. These values of σ^2_υ, σ^2_δ, and σ^2_ω imply an intracluster correlation coefficient of $\rho_\upsilon = 0.8$ and within-period correlation coefficient of $\rho_\delta = 0.1$. Importantly, our simulation results do not depend on any particular combination of these parameters values, because they rely on the internal consistency of the *ex ante* treatment effect calibration and the *ex post* estimation, conditional on a *given* set of parameter values. The only exceptions are for J and P: J must be larger enough to allow us to use the CRVE estimator (i.e., at least 40 clusters), and P must be within a reasonable range (i.e., between 0.1 and 0.9) such that there are a sufficient number of both treated clusters and control clusters.

The ANCOVA simulations in Figure 7 follow the exact same algorithm as those in Figure 2, except that after Step 3, we collapse all pre-treatment observations into a unit-specific unweighted average:

$$\overline{Y}^{FP}_{i,pre} \equiv \sum_{t\in pre} Y^{FP}_{it} \qquad \overline{Y}^{SCR}_{i,pre} \equiv \sum_{t\in pre} Y^{SCR}_{it} \qquad \overline{Y}^{0}_{i,pre} \equiv \sum_{t\in pre} Y^{0}_{it}$$

Step 4 then becomes a set of ANCOVA regressions with time fixed effects, but replacing unit fixed effects with a unit-specific pre-period control:

$$Y^{FP}_{it} = \beta + \tau^{FP} D_i + \theta \overline{Y}^{FP}_{i,pre} + \delta_t + \omega_{it} \tag{B13}$$

$$Y^{SCR}_{it} = \beta + \tau^{SCR} D_i + \theta \overline{Y}^{SCR}_{i,pre} + \delta_t + \omega_{it} \tag{B14}$$

$$Y^{0}_{it} = \beta + \tau^{0} D_i + \theta \overline{Y}^{0}_{i,pre} + \delta_t + \omega_{it} \tag{B15}$$

We estimate these ANCOVA specifications on post-treatment observations only, meaning that the treatment indicator D_i is now time-invariant. In Step 5, we collect only CRVE standard errors, and Figure 7 reports the resulting rejection rates of the ANCOVA estimator. We do not report the false rejection rates from the Y^0_i regression for the sake of brevity, and they do achieve the desired $\alpha = 0.05$ rejection rate in all cases.

An additional nuance with the ANCOVA simulations is that our simulation results are now sensitive to the intracluster correlation coefficient ρ_υ. This is because the proportion of variance that is unit-specific now affects the precision of the $\hat{\tau}$ estimator, because we have replaced the unit fixed effect (which directly controlled for this variance) with a linear control in the average pre-treatment level of the outcome variable. For a low ρ_υ, removing the unit fixed effect sacrifices little in terms of efficiency, because the between-unit differences are relatively small; for a high ρ_υ, removing the unit fixed effect can substantially reduce efficiency — enough to almost fully erode the efficiency gains of ANCOVA relative to DD. For this reason, our ANCOVA simulations in Figure 7 simulate over $m = r$, γ, and now ρ_υ. The left panel shows results for $\gamma \in \{0, 0.3, 0.5, 0.7, 0.9\}$ with $\rho_\upsilon = 0.5$; the right panel shows results for $\rho_\upsilon \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ with $\gamma = 0.5$.[8]

8. Mechanically, we vary ρ_υ by varying σ^2_υ only, and leaving σ^2_δ and σ^2_ω unchanged. The values $\sigma^2_\upsilon \in \{\frac{20}{9}, \frac{60}{7}, 20, \frac{140}{3}, 180\}$ translate to the values $\rho_\upsilon \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

B.2 Bloom et al. (2015) data

In Figure 3, we run Monte Carlo simulations using data from Bloom et al. (2015). These simulations are analogous to those described above, except that rather than simulating data, we use an actual dataset from a published panel RCT. We downloaded the paper's dataset from the *Quarterly Journal of Economics* website, and focused on the data used to estimate the paper's main results, reported in Column (1) of Table II of the paper. Consistent with the regression model that produced this result, we base our Monte Carlo analysis on the following DD specification:

$$Performance_{it} = \alpha Treat_i \times Experiment_t + \beta_t + \gamma_i + \varepsilon_{it}$$

Converting the original paper's notation to our notation, and substituting the outcome variable and fixed effects with the names of the variables in the Bloom et al. (2015) dataset, we have:

$$\underbrace{\texttt{perform1}_{it}}_{Y_{it}} = \tau D_{it} + \underbrace{\texttt{year\_week}_t}_{\delta_t} + \underbrace{\texttt{personid}_i}_{\upsilon_i} + \omega_{it}$$

We keep only units in the main sample (i.e. `expgroup` $\in \{0, 1\}$), only pre-treatment weeks of data (i.e. `year_week` < 201049), and only individuals with non-missing $\texttt{perform1}_{it}$ values for all weeks of pre-treatment data. This leaves us with a balanced panel of $J = 79$ individuals across 48 weeks.

We conduct simulations on this dataset by varying the number of pre-treatment periods (m) and the number of post-treatment periods (r). As with the simulations described above, we vary the panel length for values $m = r \in \{1, \ldots, 20\}$, iterating 10,000 simulations for each value of $m = r$. We set the parameter values $\sigma^2_\omega = 0.507$ and $\gamma = 0.233$ by regressing Y_{it} on person and week fixed effects, calculating the variance of the resulting residuals $\hat{\omega}_{it}$, and then estimating $\hat{\gamma}$ by estimating

$$\hat{\omega}_{it} = \gamma \hat{\omega}_{i(t-1)} + \xi_{it} \tag{B16}$$

Step 1: We calculate τ^{FP} and $\tau^{AR(1)}$ for each set of simulations, given $m = r$:

$$\tau^{FP} = \left(t^J_{1-\kappa} + t^J_{\alpha/2}\right) \sqrt{\left(\frac{\sigma^2_\omega}{P(1-P)J}\right)\left(\frac{m+r}{mr}\right)} \tag{B17}$$

$$\tau^{AR(1)} = (t^J_{1-\kappa} + t^J_{\alpha/2}) \sqrt{\left(\frac{1}{P(1-P)J}\right)\left[\left(\frac{m+r}{mr}\right)\sigma^2_\omega + \left(\frac{m-1}{m}\right)\ddot{\psi}^B + \left(\frac{r-1}{r}\right)\ddot{\psi}^A - 2\ddot{\psi}^X\right]} \tag{B18}$$

where

$$\ddot{\psi}^B = \frac{2\sigma^2_\omega}{(m-1)m}\sum_{z=1}^{m-1}(m-z)\gamma^z \tag{B19}$$

$$\ddot{\psi}^A = \ddot{\psi}^B \tag{B20}$$

$$\ddot{\psi}^X = \frac{\sigma^2_\omega}{m^2}\left[\sum_{z=1}^{m} z\gamma^z + \sum_{z=m+1}^{2m-1}(2m-z)\gamma^z\right] \tag{B21}$$

We denote the covariance terms as $\ddot{\psi}^B$, $\ddot{\psi}^A$, and $\ddot{\psi}^X$ to indicate that the AR(1) error assumption is a (poor) representation of the more complex covariance structure of this dataset. For both τ^{FP} and $\tau^{AR(1)}$, we calculate the critical values $t^J_{1-\kappa}$ and $t^J_{\alpha/2}$ assuming J degrees of freedom, which is consistent with applying the CRVE *ex post*, clustering at the individual level with J individuals.

Step 2: We calculate τ^{SCR} given $m = r$, by non-parametrically estimating $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$ from residuals. Appendix D.1 provides step-by-step details of this estimation algorithm. Rather than impose an AR(1) structure on the serial correlation, this method enables us to flexibly characterize the covariance structure of the Bloom et al. (2015) dataset with just three averaged parameters. This allows us to calculate τ^{SCR} as:

$$\tau^{SCR} = \left(t^J_{1-\kappa} + t^J_{\alpha/2}\right) \left\{ \left(\frac{1}{P(1-P)J}\right) \left[\left(\frac{m+r}{mr}\right) k_\sigma \sigma^2_{\hat{\omega}} + \left(\frac{m-1}{m}\right) k_B \psi^B_{\hat{\omega}} + \left(\frac{r-1}{r}\right) k_A \psi^A_{\hat{\omega}} - 2k_X \psi^X_{\hat{\omega}} \right] \right\}^{1/2} \tag{B22}$$

where

$$k_\sigma = \frac{I(m+r)^2}{2(I-1)mr}$$
$$k_B = \frac{I(m+r)^2}{2(I-1)r^2}$$
$$k_A = \frac{I(m+r)^2}{2(I-1)m^2}$$
$$k_X = 0$$

Appendix E provides a derivation of the coefficients k_σ, k_B, k_A, and k_X, and it proves that this expression for τ^{SCR} as a function of *estimated* variance-covariance parameters is equal (in expectation) to the MDE as a function of the *true* variance-covariance parameters.[9]

Step 3: For each simulation, we randomly select a range of $m+r$ consecutive weeks in the dataset. This subset of weeks will become the $(m+r)$-period panel dataset used in this particular simulation. We randomly assign treatment to PJ individuals. This involves randomly scrambling a vector of PJ ones and $(1-P)J$ zeros and assigning each individual i either a 1 indicating treatment or a 0 indicating control.[10] This allows us to construct a time-varying treatment indicator D_{it}, where $D_{it} = 1$ for all treated units in post-treatment periods only, and $D_{it} = 0$ otherwise. We then create three outcome variables by adding treatment effects to the data generated in the previous step:

$$Y^{FP}_{it} \equiv Y_{it} + \tau^{FP} D_{it} \tag{B23}$$
$$Y^{AR(1)}_{it} \equiv Y_{it} + \tau^{AR(1)} D_{it} \tag{B24}$$

9. I denotes the number of units used to estimate $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$ and $\psi^X_{\hat{\omega}}$. This is distinct from the sample size of the experiment J, however these simuations set $I = J = 79$ to include all units in the Bloom et al. (2015) dataset.

10. Our code rounds PJ to the nearest integer value, even though PJ is already an integer in our main parameterization. Note that for the τ^{FP}, $\tau^{AR(1)}$, and τ^{SCR} to be precisely calibrated, the effective $\tilde{P}$ (where $\tilde{P} = \text{round}(PJ)/J$) needs to equal the actual parameter value P.

$$Y_{it}^{SCR} \equiv Y_{it} + \tau^{SCR} D_{it} \tag{B25}$$

Step 4: We estimate the following three OLS-fixed effects regressions:

$$Y_{it}^{FP} = \beta + \tau^{FP} D_{it} + \upsilon_i + \delta_t + \omega_{it} \tag{B26}$$

$$Y_{it}^{AR(1)} = \beta + \tau^{AR(1)} D_{it} + \upsilon_i + \delta_t + \omega_{it} \tag{B27}$$

$$Y_{it}^{SCR} = \beta + \tau^{SCR} D_{it} + \upsilon_i + \delta_t + \omega_{it} \tag{B28}$$

Step 5: For each estimated $\hat{\tau}^{FP}$, $\hat{\tau}^{AR(1)}$, and $\hat{\tau}^{SCR}$, we compute CRVE standard errors, clustered at the individual level.

As with the AR(1) simulations above, we repeat Steps 3–5 10,000 times, for values of $m = r \in \{1, \ldots, 20\}$.[11] After each set of 10,000 simulations, we calculate the percent of simulations where $\hat{\tau}^{FP}$, $\hat{\tau}^{AR(1)}$, and $\hat{\tau}^{SCR}$ reject null hypothesis of $\tau = 0$ at significance level α, under CRVE standard errors. Figure 3 reports these three rejection rates on the vertical axes, with the number of pre- and post-treatment periods ($m = r$) on the horizontal axes. We can interpret these curves as realized statistical power, just as in the top row of Figure 2. We fix $\alpha = 0.05$ and $\kappa = 0.80$ across all simulations, for the reasons discussed above. Besides our arbitrary choices of $P = 0.5$, all other parameters are determined by the Bloom et al. (2015) dataset: $J = I = 79$, $\sigma_\upsilon^2 = 0.243$, $\sigma_\delta^2 = 0.146$, and $\sigma_\omega^2 = 0.507$, implying $\rho_\upsilon = 0.271$ and $\rho_\delta = 0.163$. We do not estimate β, μ_υ, or μ_δ, as these parameters are no longer relevant when simulating on top of an existing dataset.

B.3 Pecan Street data

In Figure 4, we present analogous Monte Carlo results for simulations using the Pecan Street dataset of household electricity consumption (Pecan Street (2016)). These data are publicly available (with a researcher login) at `https://dataport.pecanstreet.org/data/interactive`, and they include 699 households over 26,888 hours. As with the Bloom et al. (2015) simulations, we construct a balanced panel of households and hours, by restricting the full Pecan Street dataset to a sample of households that report non-missing, non-zero electricity consumption for every hour between January 1, 2013 and December 31, 2014. This results in a balanced panel of $J = 97$ households over 17,520 hours, which we collapse to create daily, weekly, and monthly datasets.

These Pecan Street simulations follow an algorithm identical to the Bloom et al. (2015) simulations, and we describe this algorithm in detail above. We repeat the same set of simulations four times, estimating separate rejection rates for τ^{FP}, $\tau^{AR(1)}$, and τ^{SCR}, for each of the hourly, daily, weekly, and monthly datasets. We again set $\alpha = 0.05$, $\kappa = 0.80$, and $P = 0.5$. The other relevant parameters for each dataset are:

11. As with the AR(1) simulations, we set $m = r$ only for simplicity. However, the results are very similar if fix $m = 3$ and vary $r \in \{1, \ldots, 20\}$, or vice versa.

Table B1: Pecan Street Simulation Parameters

DATASET	J	σ_v^2	σ_δ^2	σ_ω^2	γ	ρ_v	ρ_δ
Hourly	97	0.257	0.458	0.642	0.623	0.189	0.337
Daily	97	0.257	0.234	0.135	0.651	0.411	0.373
Weekly	97	0.256	0.211	0.083	0.713	0.465	0.384
Monthly	97	0.256	0.203	0.058	0.654	0.495	0.392

These values are estimated separately from each dataset used in the simulations.

B.4 Analytic Power Calculations

Figure 6 displays the results of analytic power calculations performed using the daily Pecan Street dataset. In other words, we calculate the number of units needed by applying the FP and SCR power calculation formulas by using the data to estimate $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$. For each experiment of length $m = r \in \{1, \ldots, 12\}$, we estimate the average values of $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$ over all possible panels of that length.[12] We assign half of the households to treatment ($P = 0.5$), allow for a 5 percent Type I error rate ($\alpha = 0.05$), and calibrate to 80 percent power ($\kappa = 0.80$). Finally, we rearrange Equations (3) and (D2) to calculate the number of households required to detect MDEs that range from 0 to 15 percent of baseline electricity consumption.

Figure 8 also shows the results of analytic power calculations using the SCR formula. However, instead of parameterizing Equation (5) using estimates from a dataset, we now normalize $\sigma^2_\omega = 1$ and assume an AR(1) correlation structure with $\gamma \in \{0, 0.3, 0.5, 0.7, 0.9\}$. For panel lengths $m = r \in \{1, \ldots, 100\}$, we analytically derive ψ^B, ψ^A, and ψ^X using the formulas from Equations (B3)–(B5). In the left panel, we fixe $P = 0.5$, $\alpha = 0.05$, $\kappa = 0.80$, and $J = 100$, and use Equation (5) to solve for MDE as a function of $m = r$ and γ. In the right panel, we fix $P = 0.5$, $\alpha = 0.05$, $\kappa = 0.80$, and $MDE = 1$, and rearrange Equation (5) to solve for J as a function of $m = r$ and γ.

12. We follow the algorithm outlined in Section D.1 below. For a given value of $m = r$, we consider each (consecutive) subset S of the daily Pecan Street data with length $2r$. We first residualize this subset of the data with household and day fixed effects, and we calculate $\sigma^2_{\hat{\omega},S}$ from these residuals. We then assign the first m residuals for each household to the pre-treatment period and the remaining r residuals to the post-treatment period, thereby estimating $\psi^B_{\hat{\omega}S}$, $\psi^A_{\hat{\omega}S}$, and $\psi^X_{\hat{\omega}S}$ (by averaging all pairwise covariances for subset S. Averaging $\sigma^2_{\hat{\omega}S}$, $\psi^B_{\hat{\omega}S}$, $\psi^A_{\hat{\omega}S}$, and $\psi^X_{\hat{\omega}S}$ over all subsets S, we arrive at estimates for $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$.

C Additional results

In this section, we present extensions of our simulation results from the main text. Here, we focus on short panels, as many existing randomized trials in economics contain few waves of data. We also allow m and r to vary separately in this section, to demonstrate the robustness of our results in contexts where m and r are unequal.

We begin by extending the realized power results from the middle column of Figure 2 in the main text, allowing the number of pre- and post-treatment periods to vary separately. We simulate data with serially correlated errors of varying AR(1) parameters γ, for different panel lengths, and we calibrate treatment effect sizes using the Frison and Pocock power calculation formula (Equation (3)). Figure C1 displays the results for panels with $m \in \{1, \ldots, 6\}$ and $r \in \{1, \ldots, 6\}$.

Notably, across all levels of non-zero serial correlation ($\gamma > 0$), the "naive" FP power calculation formula yields over-powered experiments for panels with either one pre-treatment or one post-treatment period. This is striking, given the number of existing experiments in economics that follow a traditional "one baseline, one follow-up" model. Given that panel data typically exhibit serial correlation, it follows that, had they relied on the FP formula to conduct *ex ante* power calculations (and had the assumed MDE been equal to the true effect), most existing experiments are likely to have been overpowered.[13] In line with the results in Figure 2, realized power decreases monotonically as panel length increases, and the FP formula becomes increasingly likely to yield underpowered experiments.

In Figure C2, we perform an analogous exercise using the Bloom et al. (2015) and Pecan Street datasets. These figures present results from the same algorithm used to produce Figures 3 and 4, except that we vary m and r separately, for $m \in \{1, \ldots, 6\}$ and $r \in \{1, \ldots, 6\}$. As with the simulated data, we find that the "naive" FP power calculation yields over-powered experiments with either one pre-treatment or one post-treatment period.

Next, rather than estimating a standard panel fixed effects model, we instead follow McKenzie (2012) and estimate treatment effects using ANCOVA methods. According to McKenzie (2012) and Frison and Pocock (1992), the analogous power calculation formula for an ANCOVA model with i.i.d. errors is:

$$MDE = \left(t_{1-\kappa} + t_{\alpha/2}\right) \sqrt{\left(\frac{\sigma_\varepsilon^2}{P(1-P)J}\right) \left[\frac{1+(r-1)\rho}{r} - \frac{m\rho^2}{1+(m-1)\rho}\right]} \tag{C1}$$

Converting this expression to our notation:

$$MDE = \left(t^J_{1-\kappa} + t^J_{\alpha/2}\right) \sqrt{\left(\frac{1}{P(1-P)J}\right) \left[(1-\theta)^2\sigma_\upsilon^2 + \frac{m+\theta^2 r}{mr}\sigma_\omega^2\right]} \tag{C2}$$

$$\text{where} \quad \theta = \frac{m\sigma_\upsilon^2}{m\sigma_\upsilon^2 + \sigma_\delta^2 + \sigma_\omega^2} \tag{C3}$$

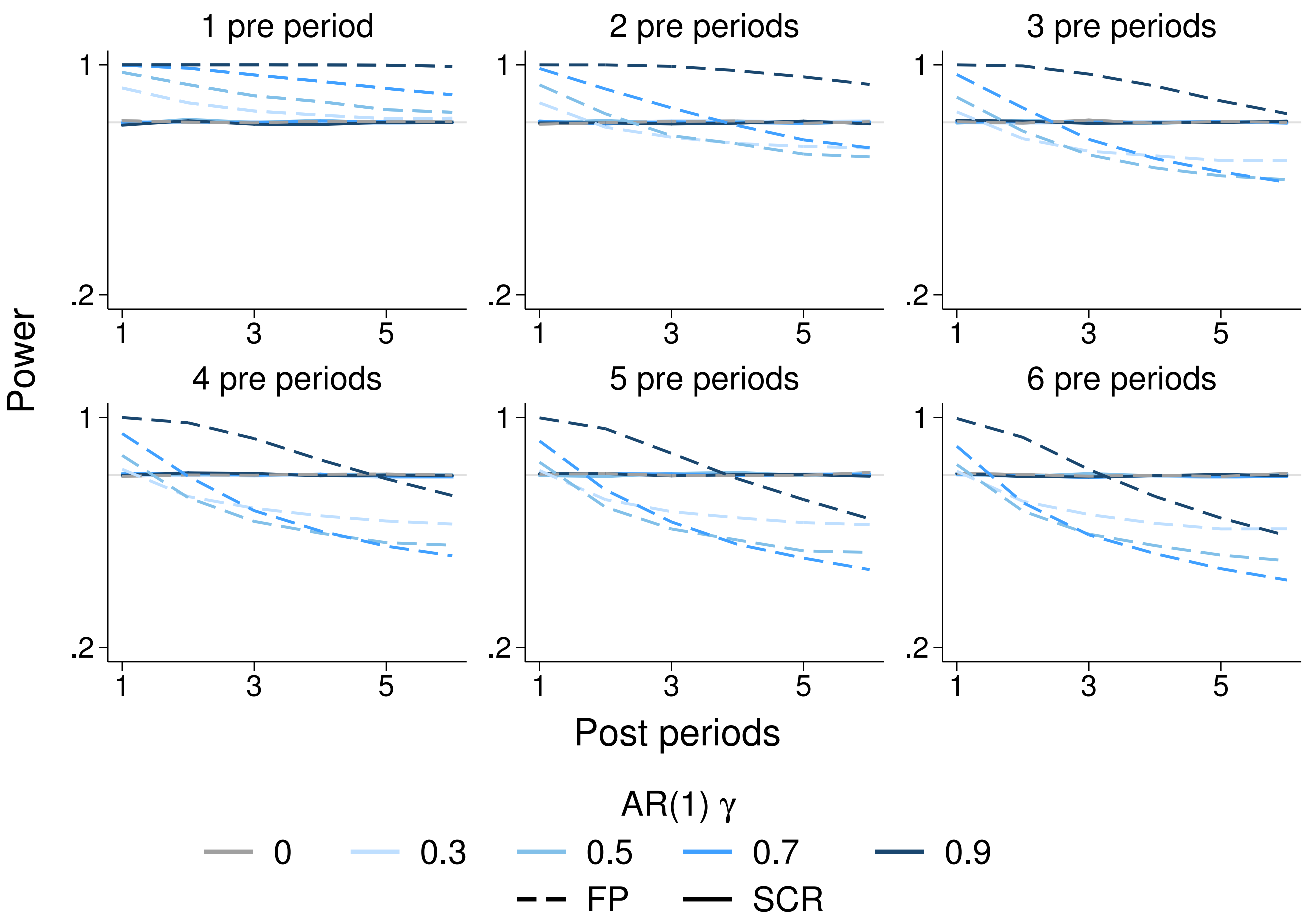
As in Figure 7, we simulate data with varying levels of AR(1) serial correlation and estimate Equation (12). We vary the number of pre-treatment and post-treatment periods separately, for

13. Technically, this also assumes that these *ex ante* power calculations parameterized the FP formula with a residual variance either calculated over a long time series, or inflated to correct for the downward bias from estimating residual variance in a short time series.

values of $m \in \{1, \ldots, 5\}$ and $r \in \{1, \ldots, 5\}$. We also vary the intracluster correlation coefficient $\rho_v \in \{0.1, 0.5, 0.9\}$. We calibrate treatment effect sizes equal to MDE in Equation (C2), and apply CRVE standard errors *ex post* (clustered by unit). Figure C3 displays the results of this exercise.

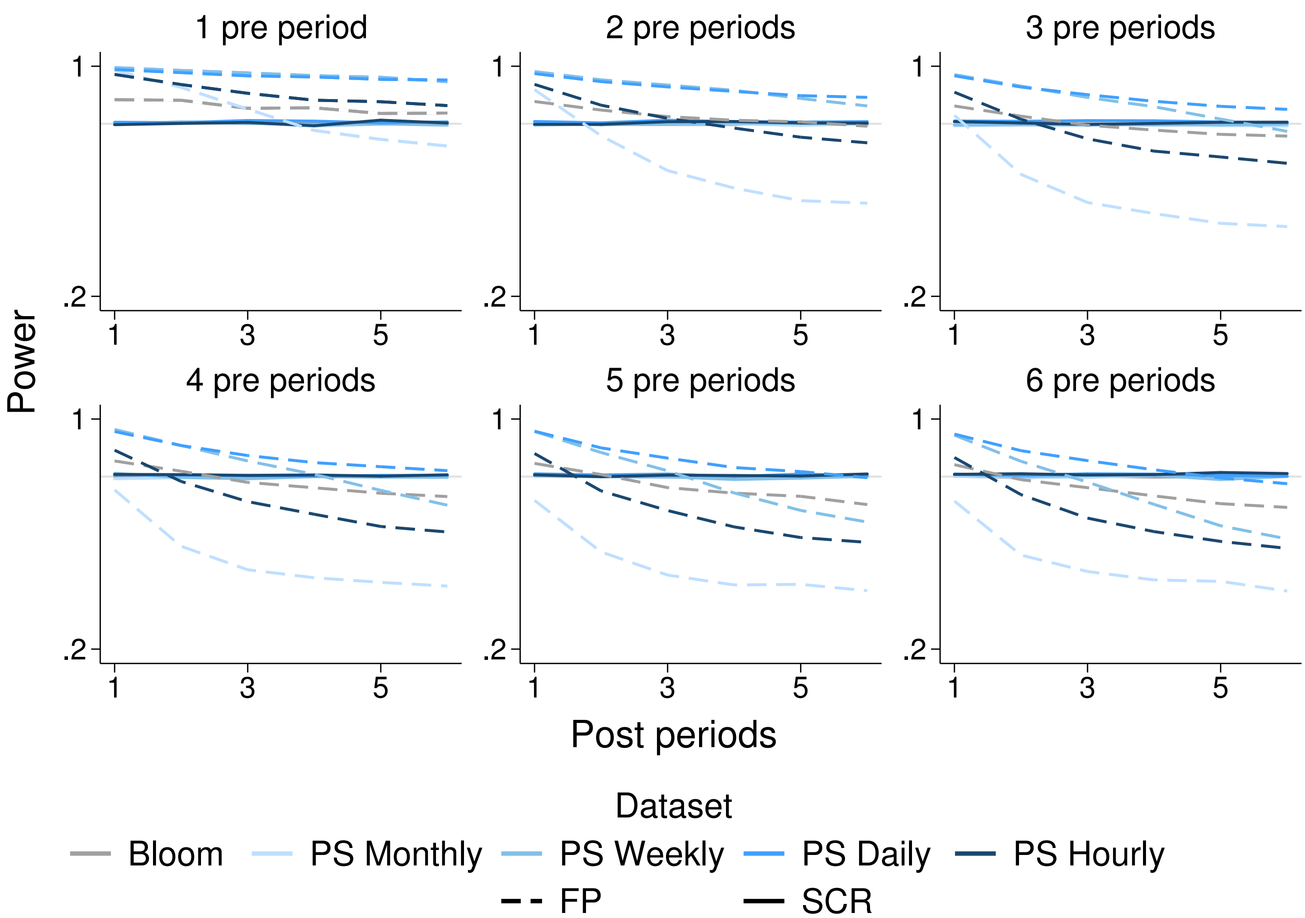
In the presence of non-zero serial correlation ($\gamma > 0$), the "naive" FP ANCOVA power calculation formula yields overpowered experiments in shorter panels, and underpowered experiments in longer panels. This is consistent with the panel DD results from Figure C1, and demonstrates that simply implementing an ANCOVA design under the i.i.d. assumption will not ensure a correctly powered experiment.

Figure C1: Power in short panels – AR(1) data



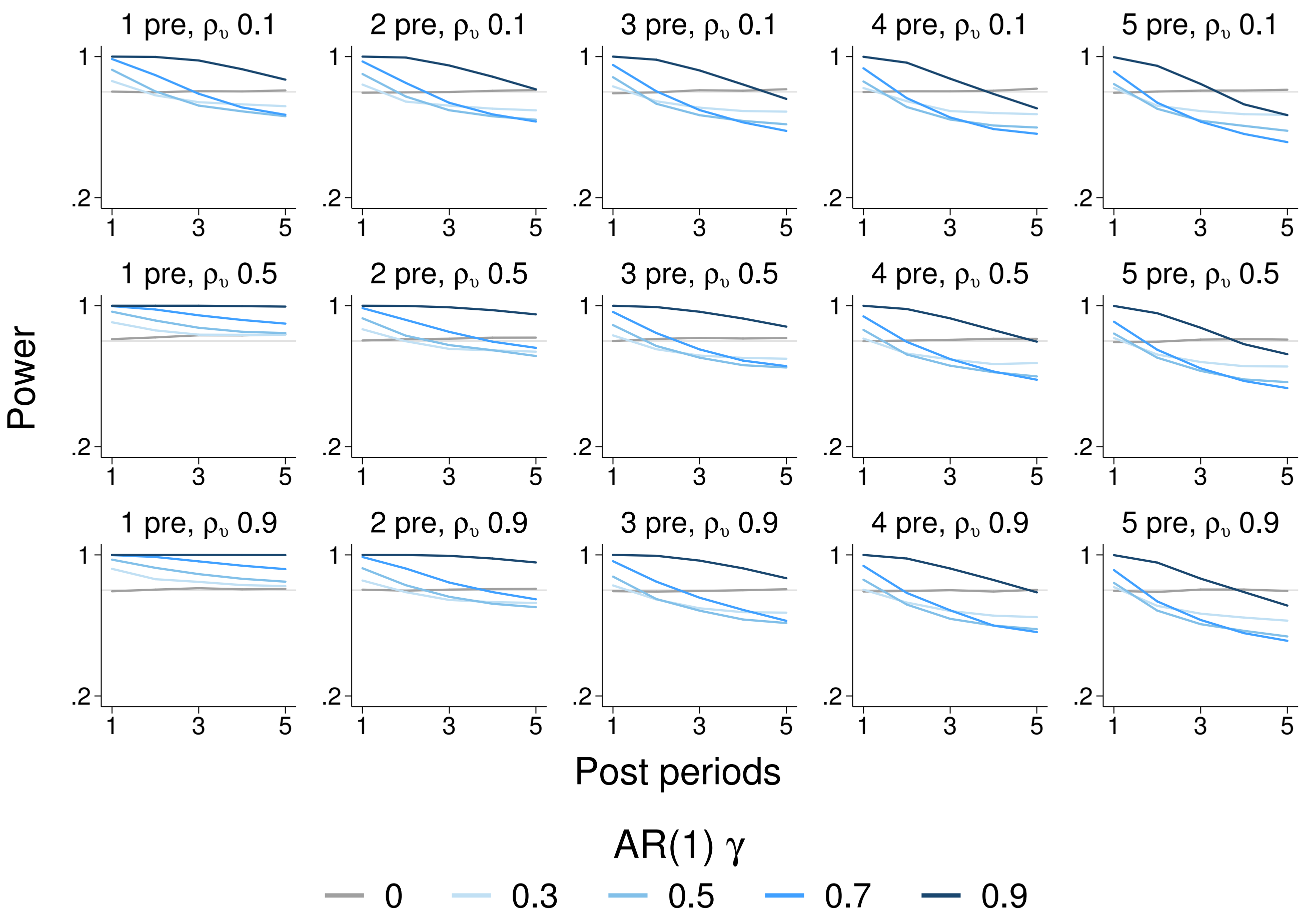
Notes: This figure displays realized power from performing power calculations using Frison and Pocock (1992)'s formula (Equation (3)) and the serial-correlation-robust formula (Equation (5)) to calibrate the treatment effect size. We cluster standard errors *ex post* in all cases, following Bertrand, Duflo, and Mullainathan (2004). We separately vary the number of the pre-treatment and post-treatment periods, where differing pre-period lengths are shown in different panels, and differing post-period lengths are shown along each panel's horizontal axis. For all cases with either one pre-treatment period or one post-treatment period, the FP formula yields over-powered experiments across the full range of positive AR(1) parameters. Experiments that follow the traditional "one baseline, one follow-up" structure will be overpowered, having calibrated an excessively large sample size. As the number of pre-treatment periods increases, power decreases monotonically for all $\gamma > 0$. At the same time, the serial-correlation-robust formula is properly powered in all cases.

Figure C2: Power in short panels – Real data



Notes: This figure displays realized power from performing power calculations using Frison and Pocock (1992)'s formula (Equation (3)) and the serial-correlation-robust formula (Equation (5)) to calibrate the treatment effect size. We cluster standard errors *ex post* in all cases, following Bertrand, Duflo, and Mullainathan (2004). We separately vary the number of the pre-treatment and post-treatment periods, where differing pre-period lengths are shown in different panels, and differing post-period lengths are shown along each panel's horizontal axis. We show results for five different datasets: the Bloom et al. (2015) data, as well as monthly, weekly, daily, and hourly Pecan Street data. For nearly all cases with either one pre-treatment period or one post-treatment period, the FP formula yields over-powered experiments. Experiments that follow the traditional "one baseline, one follow-up" structure will be overpowered, having calibrated an excessively large sample size. As the number of pre-treatment periods increases, power decreases monotonically for each dataset. By contrast, the serial-correlation-robust formula is properly powered in all cases.

Figure C3: Traditional ANCOVA methods fail to achieve desired power



Notes: This figure displays realized power from performing power calculations using Frison and Pocock (1992)'s ANCOVA formula (Equation (C2)) to calibrate the treatment effect size. We cluster standard errors ex post in all cases, following Bertrand, Duflo, and Mullainathan (2004), and estimate treatment effects using the ANCOVA estimating Equation (12). We separately vary the number of pre-treatment periods (by column), the number of post-treatment periods (along each panel's horizontal axis), and the intra-cluster correlation of the pre-period, the length of the post-period, and the intracluster correlation coefficient (by by row). As with Figure C1, applying the FP ANCOVA formula in the presence of serial correlation will lead to incorrectly experiments, and experiments with either one pre- or one post-treatment period will be overpowered.

D A practical guide to power calculations

In this section, we address several practical considerations when conducting power calculations. Most of these challenges involve variance and covariance parameters that must be either estimated or assumed in order to operationalize a power calculation formula. We also outline steps for estimating power calculations via simulation, which is our preferred method but requires a representative pre-existing dataset.

D.1 Analytical power calculations

The most challenging aspect of analytical power calculations is parameterizing the variance and (if applicable) covariance terms that characterize the data's error structure. In the absence of a representative pre-existing dataset, researchers may struggle to even guess the order of magnitude of the error variance, let alone generate a precise estimate of this key parameter. Our theoretical results demonstrate that the error covariance structure is likewise key to determining the statistical power of panel RCTs. As in the paper, we denote the true parameters governing the data generating process as σ^2_ω, ψ^B, ψ^A, and ψ^X. We define $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$ to be the parameters that characterize the *residuals* (rather than real errors). If researchers do have access to a representative dataset when performing *ex ante* power calculations, they can directly estimate $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$, and use these values to parameterize power calculations. Section E.2 proves that researchers can recover the true MDE using these residual-based parameters. Nevertheless, this process is not trivial, for several reasons.

First, while the idiosyncratic variance σ^2_ω is a population parameter, the three ψ parameters are functions of both the full covariance structure of the population and the specific values of m and r.[14] For a given population and serially correlated outcome variable, experiments with small m and r are likely to exhibit larger ψ^B, ψ^A, and ψ^X parameters than experiments with large m and r. This is because as the number of pre-treatment (post-treatment) periods increases, ψ^B (ψ^A) averages across covariances of time periods that are farther apart. For example, compare ψ^B with $m = 3$ vs. $m = 30$, for an outcome with a covariance structure where adjacent periods are more positively correlated than distant periods. For $m = 3$, ψ^B averages $m(m-1)/2 = 3$ pairwise covariances, 2 of which are for adjacent periods; for $m = 30$, ψ^B averages $m(m-1)/2 = 435$ pairwise covariances, only 29 of which are for adjacent periods. Because ψ^X expands with both m and r, it attenuates relatively faster than ψ^B and ψ^A as panel length grows.

Second, estimating $\text{Cov}(\hat{\omega}_{it}, \hat{\omega}_{is})$ using residuals from an existing dataset is fundamentally impossible, given that each dataset contains only one realization of (Y_{it}, Y_{is}). However, researchers may treat the $(I \times 1)$ vectors of residuals $(\vec{\hat{\omega}}_t, \vec{\hat{\omega}}_s)$ as I draws from the distributions of residuals for periods (t,s) and estimate these distributions' covariance. The resulting estimates, which we denote $\tilde{\sigma}^2_{\hat{\omega}}$, $\tilde{\psi}^B_{\hat{\omega}}$, $\tilde{\psi}^A_{\hat{\omega}}$, and $\tilde{\psi}^X_{\hat{\omega}}$, are unbiased estimators of $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$.[15]

Third, if the representative dataset contains a long time series, the residual variance and covariance structure may change throughout the time series. This means if researchers estimate $\tilde{\sigma}^2_{\hat{\omega}}$, $\tilde{\psi}^B_{\hat{\omega}}$, $\tilde{\psi}^A_{\hat{\omega}}$, and $\tilde{\psi}^X_{\hat{\omega}}$ by averaging across the full time series, these estimated parameters may be

14. Deriving the residual-based parameters $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$ introduces an additional complexity, as these residual-based parameters are defined by the number of pre-treatment periods (m), post-treatment periods (r) and cross-sectional units (I) used to produce these residuals.

15. Appendix E.1 proves that $\text{E}[\tilde{\sigma}^2_{\hat{\omega}} \mid \mathbf{X}] = \sigma^2_{\hat{\omega}}$, $\text{E}[\tilde{\psi}^B_{\hat{\omega}} \mid \mathbf{X}] = \psi^B_{\hat{\omega}}$, $\text{E}[\tilde{\psi}^A_{\hat{\omega}} \mid \mathbf{X}] = \psi^A_{\hat{\omega}}$, and $\text{E}[\tilde{\psi}^X_{\hat{\omega}} \mid \mathbf{X}] = \psi^X_{\hat{\omega}}$.

less representative than if they were estimated from just the end of the time series.[16] Furthermore, because the residual variance is not a function of panel length, it may be tempting to estimate $\tilde{\sigma}^2_{\hat{\omega}}$ using a long vector of residuals, while estimating $\tilde{\psi}^B_{\hat{\omega}}$, $\tilde{\psi}^A_{\hat{\omega}}$, and $\tilde{\psi}^X_{\hat{\omega}}$ using only residuals within an $(m+r)$-period range. In a time series where the variance-covariance structure is changing, this would produce $\tilde{\psi}_{\hat{\omega}}$ estimates that are inconsistent with $\tilde{\sigma}^2_{\hat{\omega}}$.

Fourth, while $\tilde{\sigma}^2_{\hat{\omega}}$, $\tilde{\psi}^B_{\hat{\omega}}$, $\tilde{\psi}^A_{\hat{\omega}}$, and $\tilde{\psi}^X_{\hat{\omega}}$ are unbiased estimators of $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$, they are **not** unbiased estimators of σ^2_{ω}, ψ^B, ψ^A, and ψ^X. This is because the residuals from the regression $Y_{it} = \upsilon_i + \delta_t + \omega_{it}$ will have a variance less than the parameter σ^2_{ω} from the data generating process, by the properties of linear projection. In addition, when they are estimated using residuals from shorter panels, $\tilde{\sigma}^2_{\hat{\omega}}$, $\tilde{\psi}^B_{\hat{\omega}}$, $\tilde{\psi}^A_{\hat{\omega}}$, and $\tilde{\psi}^X_{\hat{\omega}}$ have a more severe bias, but these estimates converge to their true values (i.e., σ^2_{ω}, ψ^B, ψ^A, and ψ^X) as the panel length used to the estimate these residuals increases.[17] Importantly, for the purposes of power calculations using the SCR formula, we *can* recover an unbiased estimate of the minimum detectable effect with the true parameters using our parameter estimates. That is, $MDE^{est}(\sigma^2_{\hat{\omega}}, \psi^B_{\hat{\omega}}, \psi^A_{\hat{\omega}}, \psi^X_{\hat{\omega}}) = MDE(\sigma^2_{\omega}, \psi^B, \psi^A, \psi^X)$. As Section E.1 shows, $\mathrm{E}[\tilde{\sigma}^2_{\hat{\omega}} \mid \mathbf{X}] = \sigma^2_{\hat{\omega}}$, $\mathrm{E}[\tilde{\psi}^B_{\hat{\omega}} \mid \mathbf{X}] = \psi^B_{\hat{\omega}}$, $\mathrm{E}[\tilde{\psi}^A_{\hat{\omega}} \mid \mathbf{X}] = \psi^A_{\hat{\omega}}$, and $\mathrm{E}[\tilde{\psi}^X_{\hat{\omega}} \mid \mathbf{X}] = \psi^X_{\hat{\omega}}$. Combining these two proofs suggests that $MDE^{est}(\mathrm{E}[\tilde{\sigma}^2_{\hat{\omega}} \mid \mathbf{X}], \mathrm{E}[\tilde{\psi}^B_{\hat{\omega}} \mid \mathbf{X}], \mathrm{E}[\tilde{\psi}^A_{\hat{\omega}} \mid \mathbf{X}], \mathrm{E}[\tilde{\psi}^X_{\hat{\omega}} \mid \mathbf{X}]) = MDE(\sigma^2_{\omega}, \psi^B, \psi^A, \psi^X)$. Therefore, for values of $\tilde{\sigma}^2_{\hat{\omega}}$, $\tilde{\psi}^B_{\hat{\omega}}$, $\tilde{\psi}^A_{\hat{\omega}}$, and $\tilde{\psi}^X_{\hat{\omega}}$ estimated from a pre-existing dataset with I cross-sectional units:

$$\begin{aligned} MDE^{est} &= \left(t^J_{1-\kappa} + t^J_{\alpha/2}\right) \left\{ \left(\frac{1}{P(1-P)J}\right) \left[\left(\frac{m+r}{mr}\right) k_\sigma \, \mathrm{E}\left[\tilde{\sigma}^2_{\hat{\omega}} \mid \mathbf{X}\right] + \left(\frac{m-1}{m}\right) k_B \, \mathrm{E}\left[\tilde{\psi}^B_{\hat{\omega}} \mid \mathbf{X}\right] \right.\right. \\ &\qquad \left.\left. + \left(\frac{r-1}{r}\right) k_A \, \mathrm{E}\left[\tilde{\psi}^A_{\hat{\omega}} \mid \mathbf{X}\right] - 2k_X \, \mathrm{E}\left[\tilde{\psi}^X_{\hat{\omega}} \mid \mathbf{X}\right] \right] \right\}^{1/2} \qquad \text{(D1)} \\ &= \left(t^J_{1-\kappa} + t^J_{\alpha/2}\right) \left\{ \left(\frac{1}{P(1-P)J}\right) \left[\left(\frac{m+r}{mr}\right) k_\sigma \sigma^2_{\hat{\omega}} + \left(\frac{m-1}{m}\right) k_B \psi^B_{\hat{\omega}} \right.\right. \\ &\qquad \left.\left. + \left(\frac{r-1}{r}\right) k_A \psi^A_{\hat{\omega}} - 2k_X \psi^X_{\hat{\omega}} \right] \right\}^{1/2} \qquad \text{(D2)} \end{aligned}$$

where

$$\begin{aligned} k_\sigma &= \frac{I(m+r)^2}{2(I-1)mr} \\ k_B &= \frac{I(m+r)^2}{2(I-1)r^2} \\ k_A &= \frac{I(m+r)^2}{2(I-1)m^2} \\ k_X &= 0 \end{aligned}$$

16. Of course, if the researcher knows that certain parts of her data are more likely to represent the experimental timeframe and data, it would be wise to perform power calculations on this subset alone.

17. The estimated residuals include both the true idiosyncratic error, ω_{it}, and (attenuating) fixed-effect estimation error. Although both sets fixed effects, υ_i and δ_t, are unbiased and consistent in T and I, respectively, error in estimating these parameters will always yield residuals that are smaller on average, biasing the estimation of these parameters. The estimation error and resulting biases decrease in T and I.

and the expectation of parameters are taken over subsets of the dataset, as described in the next point. Appendix E.2 proves that Equations (D1) and (D2) are equivalent, and we derive the above expressions for the coefficients k_σ, k_B, k_A, and k_X in Appendix E.2.

Fifth, because the estimated variance-covariance terms enter the power calculation under a radical, researchers must be conscious of Jensen's Inequality. If the researcher is estimating $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$ by taking the expectation of $\tilde{\sigma}^2_{\hat{\omega}}$, $\tilde{\psi}^B_{\hat{\omega}}$, $\tilde{\psi}^A_{\hat{\omega}}$, and $\tilde{\psi}^X_{\hat{\omega}}$ across a range of $(m+r)$-period subsets, then the correct calculation is:

$$MDE^{est}\left(\mathrm{E}\left[\tilde{\sigma}^2_{\hat{\omega}} \mid \mathbf{X}\right], \mathrm{E}\left[\tilde{\psi}^B_{\hat{\omega}} \mid \mathbf{X}\right], \mathrm{E}\left[\tilde{\psi}^A_{\hat{\omega}} \mid \mathbf{X}\right], \mathrm{E}\left[\tilde{\psi}^X_{\hat{\omega}} \mid \mathbf{X}\right]\right), \text{ not } \mathrm{E}\left[MDE^{est}\left(\tilde{\sigma}^2_{\hat{\omega}}, \tilde{\psi}^B_{\hat{\omega}}, \tilde{\psi}^A_{\hat{\omega}}, \tilde{\psi}^X_{\hat{\omega}}\right) \mid \mathbf{X}\right].$$

Similarly, if Equation (5) is rearranged as a function of κ, it becomes convex in the variance-covariance parameters, and the correct calculation is:

$$\kappa\left(\mathrm{E}\left[\tilde{\sigma}^2_{\hat{\omega}} \mid \mathbf{X}\right], \mathrm{E}\left[\tilde{\psi}^B_{\hat{\omega}} \mid \mathbf{X}\right], \mathrm{E}\left[\tilde{\psi}^A_{\hat{\omega}} \mid \mathbf{X}\right], \mathrm{E}\left[\tilde{\psi}^X_{\hat{\omega}} \mid \mathbf{X}\right]\right), \quad \text{not } \mathrm{E}\left[\kappa\left(\tilde{\sigma}^2_{\hat{\omega}}, \tilde{\psi}^B_{\hat{\omega}}, \tilde{\psi}^A_{\hat{\omega}}, \tilde{\psi}^X_{\hat{\omega}}\right) \mid \mathbf{X}\right].$$

When solving for sample size J, Equation (5) becomes linear in variance-covariance parameters, meaning that Jensen's Inequality does not affect the estimate of $J\left(\sigma^2_{\hat{\omega}}, \psi^B_{\hat{\omega}}, \psi^A_{\hat{\omega}}, \psi^X_{\hat{\omega}}\right)$.

In light of each of these issues, we recommend the following algorithm for estimating the MDE using a pre-existing panel dataset:[18]

1. Determine all feasible ranges of experiments with $(m+r)$ periods, given the number of time periods in the pre-existing dataset. For example, if this dataset contains 100 time periods indexed $t = \{1, \ldots, 100\}$, and $m = 5$ and $r = 6$, then there are 90 feasible ranges for an experiment with $(m+r) = 11$ periods (i.e., beginning in periods $t = \{1, \ldots, 90\}$).

2. For each feasible range S:

   (a) Regress the outcome variable on unit and time-period fixed effects, $Y_{it} = \upsilon_i + \delta_t + \omega_{it}$, and store the residuals. (This regression includes all I available cross-sectional units, but only time periods with the specific range S.[19])

   (b) Calculate the variance of the stored residuals, and save as $\tilde{\sigma}^2_{\hat{\omega},S}$.

   (c) For each pair of pre-treatment periods (i.e., the first m periods in range S), calculate the covariance between these periods' residuals. Take an unweighted average of these $m(m-1)/2$ covariances, and save as $\tilde{\psi}^B_{\hat{\omega},S}$.

   For example, if $m = 4$, $r = 2$, and range S begins in period $t = 1$, sum $\mathrm{Cov}(\omega_{i1}, \omega_{i2})$, $\mathrm{Cov}(\omega_{i1}, \omega_{i3})$, $\mathrm{Cov}(\omega_{i1}, \omega_{i4})$, $\mathrm{Cov}(\omega_{i2}, \omega_{i3})$, $\mathrm{Cov}(\omega_{i2}, \omega_{i4})$, and $\mathrm{Cov}(\omega_{i3}, \omega_{i4})$, and divide by $m(m-1)/2 = 6$.

   (d) For each pair of post-treatment periods (i.e., the last r periods in range S), calculate the covariance between these periods' residuals. Take an unweighted average of these $r(r-1)/2$ covariances, and save as $\tilde{\psi}^A_{\hat{\omega},S}$.

18. Our accompanying software packages implement this algorithm using the programs `pc_dd_covar` (in STATA) and `PCDDCovar` (in `R`).

19. This bears no relationship to the sample size J units to be included in the power calculation. Assuming that all I units in the pre-existing dataset represent the population to be included in the randomization, estimating the variance and covariances using all available units will provide the best estimates of $\tilde{\sigma}^2_{\hat{\omega}}$, $\tilde{\psi}^B_{\hat{\omega}}$, $\tilde{\psi}^A_{\hat{\omega}}$, and $\tilde{\psi}^X_{\hat{\omega}}$ (by the weak law of large numbers).

For example, if $m = 4$, $r = 2$, and range S begins in period $t = 1$, $\tilde{\psi}^{A}_{\hat{\omega},S}$ is the average of a single post-period covariance, $\text{Cov}(\omega_{i5}, \omega_{i6})$.

(e) For each pair of pre- and post-treatment periods (i.e. the first m and the last r periods in range S), calculate the covariance between these periods' residuals. Take an unweighted average of these mr covariances, and save as $\tilde{\psi}^{X}_{\hat{\omega},S}$.

For example, if $m = 4$, $r = 2$, and range S begins in period $t = 1$, sum $\text{Cov}(\omega_{i1}, \omega_{i5})$, $\text{Cov}(\omega_{i1}, \omega_{i6})$, $\text{Cov}(\omega_{i2}, \omega_{i5})$, $\text{Cov}(\omega_{i2}, \omega_{i6})$, $\text{Cov}(\omega_{i3}, \omega_{i5})$, $\text{Cov}(\omega_{i3}, \omega_{i6})$, $\text{Cov}(\omega_{i4}, \omega_{i5})$, and $\text{Cov}(\omega_{i4}, \omega_{i6})$, and divide by $mr = 8$.

3. Calculate the average of $\tilde{\sigma}^2_{\hat{\omega},S}$, $\tilde{\psi}^{B}_{\hat{\omega},S}$, $\tilde{\psi}^{A}_{\hat{\omega},S}$, and $\tilde{\psi}^{X}_{\hat{\omega},S}$ across all ranges S, deflating $\tilde{\sigma}^2_{\hat{\omega},S}$ by $\frac{IT-1}{IT}$, and $\tilde{\psi}^{B}_{\hat{\omega},S}$, $\tilde{\psi}^{A}_{\hat{\omega},S}$, and $\tilde{\psi}^{X}_{\hat{\omega},S}$ by $\frac{I-1}{I}$. These averages are equal in expectation to $\sigma^2_{\hat{\omega}}$, $\psi^{B}_{\hat{\omega}}$, $\psi^{A}_{\hat{\omega}}$, and $\psi^{X}_{\hat{\omega}}$.

4. Plug these values into Equation (D2) to produce MDE^{est}.

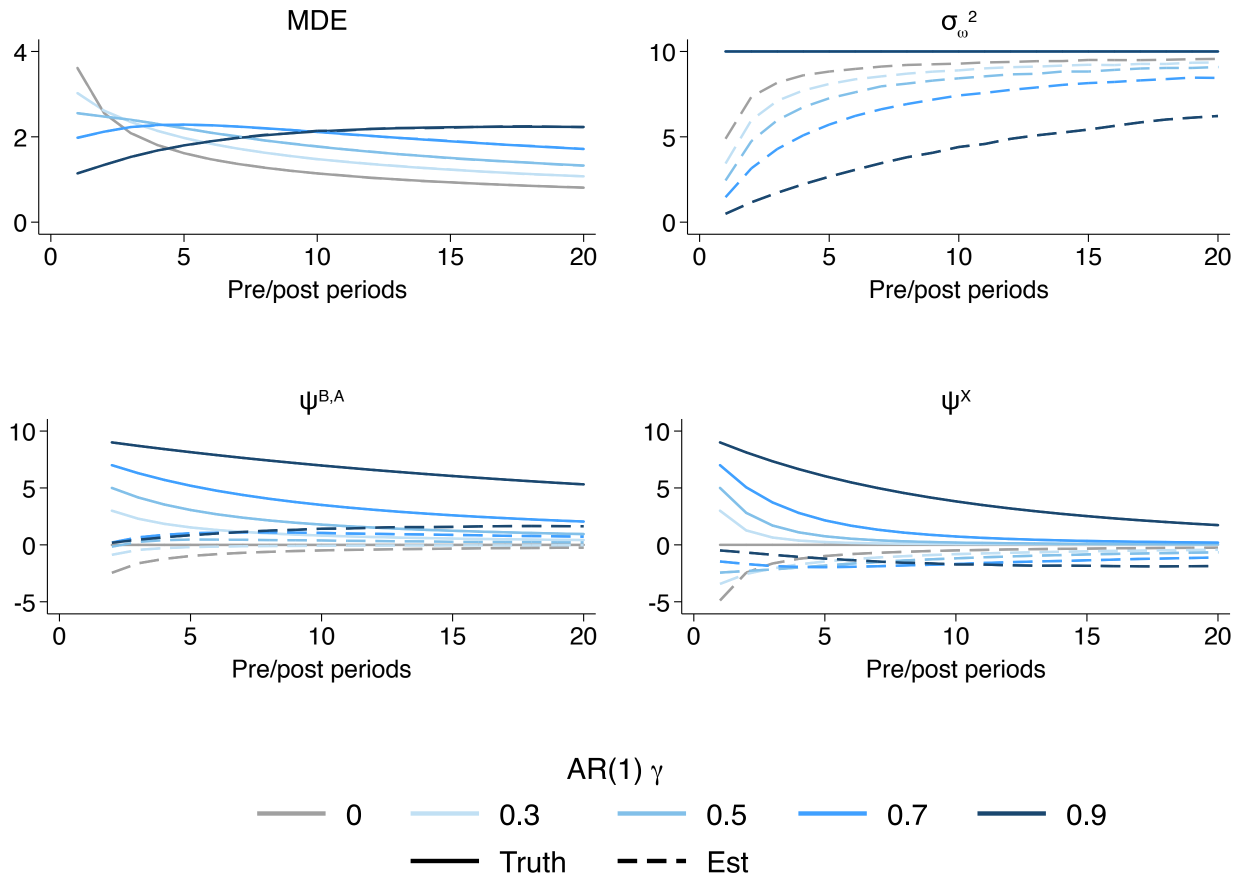
Figure D1 shows the difference between true and estimated variance-covariance parameters in AR(1) data. In particular, we show true parameter values of σ^2_{ω}, ψ^B, ψ^A, and ψ^X alongside estimated values of these same parameters, calculated according to the procedure outlined above. As expected, $\sigma^2_{\hat{\omega}}$ is biased downwards relative to σ^2_{ω}, but converges towards this value as the panel length increases. This convergence is slower for larger AR(1) parameters, as highly serially correlated errors make it harder to identify the unit fixed effects. Similarly, while the true ψ^X is positive across all panel lengths, $\psi^{X}_{\hat{\omega}}$ is negative everywhere, and ψ^B and ψ^A also differ from their estimated counterparts. Despite the differences between the true parameters and their estimated values, Section E.2 proves that we can recover the MDE based on true underlying parameters using residual-based parameters. In conjunction with the fact that we can estimate the residual-based parameters from real data, this confirms that researchers can use estimated parameters to calibrate power calculations.

Figure D2 uses the Bloom et al. (2015) dataset to present an analogous comparison between actual vs. estimated σ^2_{ω}, $\psi^{B,A}$, and ψ^X parameters. Here, as in Figure D1, the dotted lines estimate $\sigma^2_{\hat{\omega}}$, $\psi^{B,A}_{\hat{\omega}}$, and $\psi^{X}_{\hat{\omega}}$ using the above algorithm. However, unlike with simulated AR(1) datasets, the "true" parameters of the Bloom et al. (2015) data generating process are unknown. We estimate these "true" values using residuals from the full 48-period time series, which minimizes the fixed effect estimation error that biases $\sigma^2_{\hat{\omega}}$, $\psi^{B}_{\hat{\omega}}$, $\psi^{A}_{\hat{\omega}}$, and $\psi^{X}_{\hat{\omega}}$ in short panels.[20] This reveals a very similar pattern: "subsetted" $\sigma^2_{\hat{\omega}}$, $\psi^{B}_{\hat{\omega}}$, $\psi^{A}_{\hat{\omega}}$, and $\psi^{X}_{\hat{\omega}}$ estimates are systematically biased downward, but converge to their "full time-series" (i.e. closer to "true") values as panel length increases. As in Figure D1, we show that both sets of estimated variance-covariance parameters yield (virtually) identical MDEs, as long as Equation (5) uses estimated parameters that are internally consistent.

Figure D3 replicates Figure D2 for all four Pecan Street datasets. We see that while the estimated values $\sigma^2_{\hat{\omega}}$, $\psi^{B}_{\hat{\omega}}$, $\psi^{A}_{\hat{\omega}}$, and $\psi^{X}_{\hat{\omega}}$ differ across different levels of aggregation, they follow the same pattern. The subsetted estimates are biased downward, but appear to converge to the full time series estimates (i.e. closer to truth) as panel length increases. In all four cases, the MDE is (virtually) identical when calculated using either all full time series estimates or all subsetted estimates.

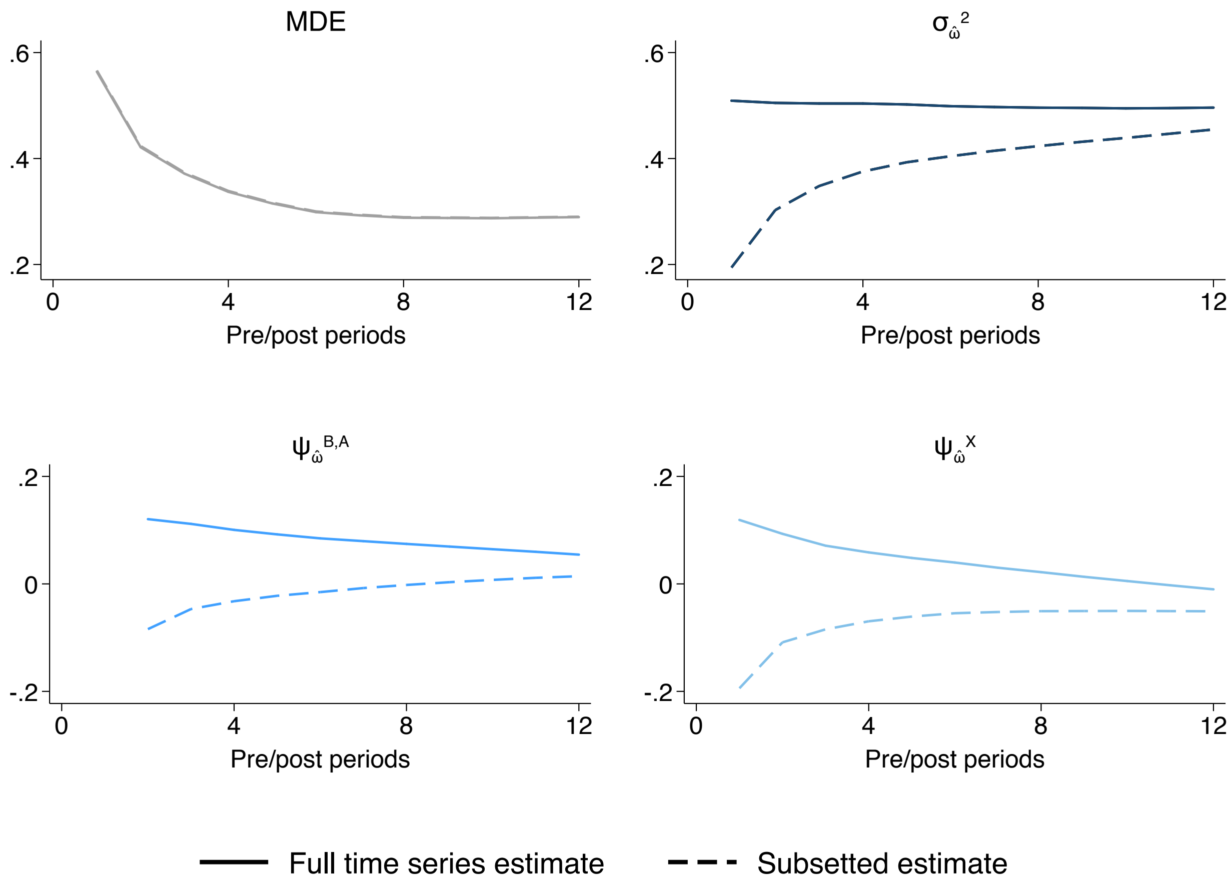
20. These $\sigma^2_{\hat{\omega}}$, $\psi^{B}_{\hat{\omega}}$, $\psi^{A}_{\hat{\omega}}$, and $\psi^{X}_{\hat{\omega}}$ estimates (represented by solid lines in Figure D2) result from the same algorithm as detailed above, except omitting Step 2(a) and estimating a single 48-period set of residuals in Step 1. This provides the closest possible approximation to the "true" variance-covariance structure of these data, and hence the most apples-to-apples comparison to Figure D1.

Figure D1: Actual vs. estimated parameters – AR(1) data



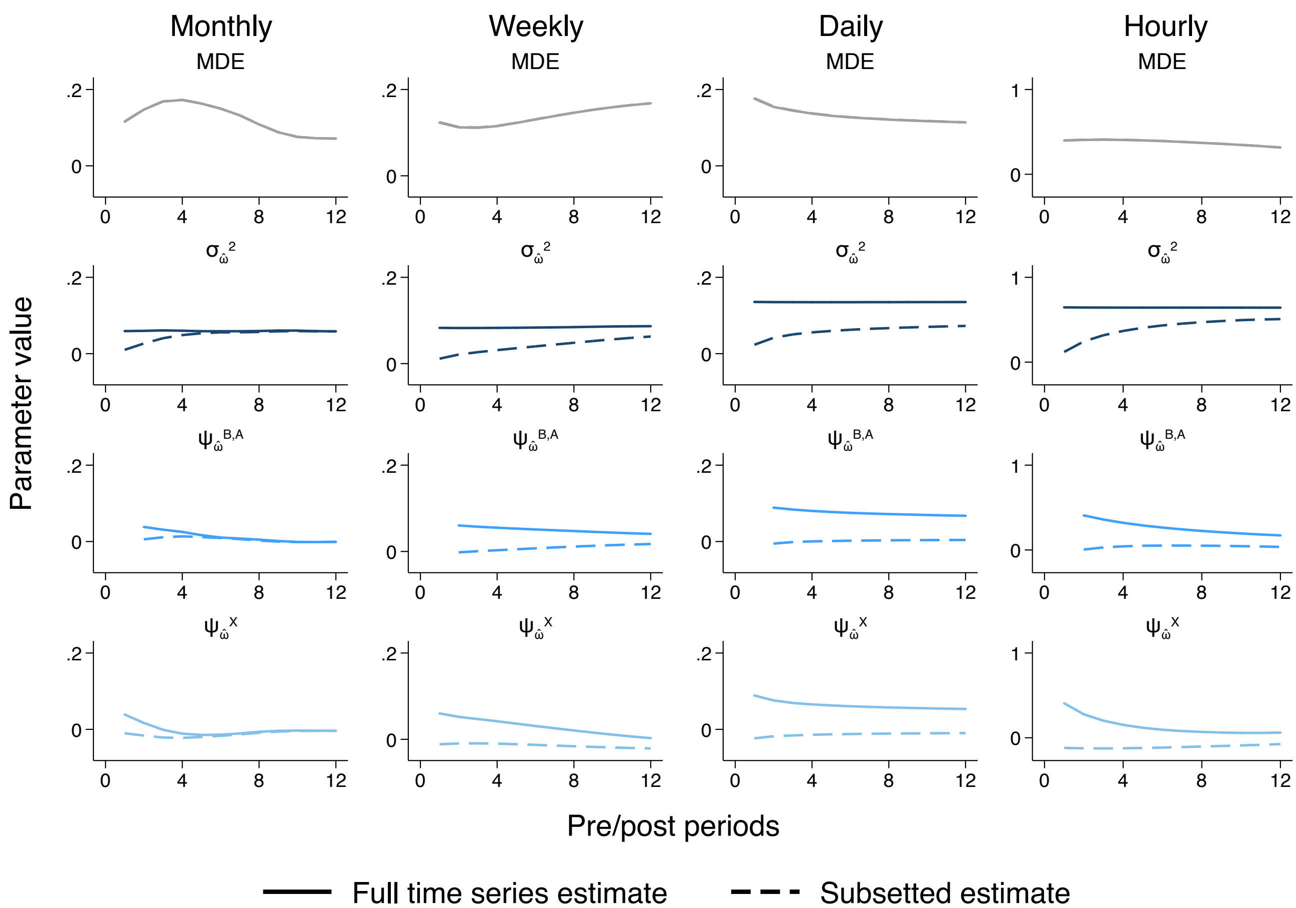
Notes: This figure displays the difference between the true residual variance (σ^2_ω), average pre (post)-period covariance ($\psi^{B,A}$), and average cross-period covariance (ψ^X), and their estimated counterparts over varying panel lengths. This figure also shows the resulting minimum detectable effect (MDE), calculated using the serial-correlation-robust power calculation formula (Equation (5)). These parameters and estimates come from a simulated datasets with AR(1) errors, generated identically to those presented in Figure 2. True parameters are displayed with solid lines (with ψ terms derived analytical using Equations (B3)–(B5)), and estimates are displayed with dashed lines (estimated according to the algorithm described above). As expected, the true σ^2_ω is constant across panel lengths, while $\psi^{B,A}$ declines with the number of pre (post) periods, and ψ^X declines more quickly than $\psi^{B,A}$ as the panel length increases. The higher the AR(1) parameter, the larger are the ψ terms. The estimated parameters behave quite differently from their true counterparts. In short panels, $\sigma^2_{\hat{\omega}}$ is biased downwards, because the regression model's individual fixed effects are inconsistently estimated, and capture more of the true error variance than they should explain in expectation. This has the effect of reducing the estimated covariances $\psi^{B,A}_{\hat{\omega}}$, which scale with $\sigma^2_{\hat{\omega}}$. At the same time, $\psi^X_{\hat{\omega}}$ is mechanically negative, as the estimated fixed effects yield residuals that are negatively correlated within individuals across pre/post-treatment time periods (and some of the variation that should be captured by $\sigma^2_{\hat{\omega}}$ is instead loaded on to the $\psi_{\hat{\omega}}$ terms). Note that as the panel length increases, the $\psi_{\hat{\omega}}$ terms converge to the true ψ values. Importantly, despite the fact that the estimated parameters are different from the true parameters, these differences offset such that both estimated and real parameters result in the same MDEs, as demonstrated by the top left panel.

Figure D2: Estimated parameters - Bloom et al. (2015) data



Notes: This figure shows two different methods for estimating variance parameters, applied to the Bloom et al. (2015) data, and also depicts the resulting minimum detectable effect (MDE) resulting from both methods. The solid lines show parameters calculated by running a regression of the outcome variable on unit and time period fixed effects, estimated on the entire time series of data, and then using the residuals from this regression to calculate $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$ for the average panel of length $m+r$. We then plug these estimates into Equation (5) to calculate the minimum detectable effect. The dashed lines show parameters estimated using the procedure described above, where rather than use residuals from the full time series, we subset the dataset into shorter panels of length $m+r$, calculate the parameters using residuals only from this subset, and average across all possible subsets to arrive at $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$. We calculate the MDE by plugging these estimates into Equation (D2). Note that these variance-covariance estimates converge as the panel length increases. Both procedures yield (virtually) identical MDEs, even though the underlying parameter estimates differ substantially.

Figure D3: Estimated parameters - Pecan Street data



Notes: This figure shows two different methods for estimating variance parameters, applied to the Pecan Street data at the four levels of aggregation presented in the main text, and also depicts the resulting minimum detectable effect (MDE) resulting from both methods. Note that the y axis scale differs between the hourly data and the other three datasets; this is because the degree of residual variation left in the hourly data after removing time and individual fixed effects is much greater than in the other datasets, which is to be expected. The solid lines show parameters calculated by running a regression of the outcome variable on unit and time period fixed effects, estimated on the entire time series of data, and then using the residuals from this regression to calculate $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$ for the average panel of length $m + r$. We then plug these estimates into Equation (5) to calculate the minimum detectable effect. By contrast, the dashed lines show parameters estimated using the procedure described above, where rather than use residuals from the full time series, we subset the dataset into shorter panels of length $m + r$, calculate the parameters using residuals only from this subset, and average across all possible subsets to arrive at $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$. We calculate the MDE by plugging these estimates into Equation (D2). Note that these variance-covariance estimates converge as the panel length increases. Both procedures yield (virtually) identical MDEs, even though the two procedures' method for estimating the underlying parameters differ substantially.

Two additional nuances that arise during analytical power calculations are worth noting. First, the critical values $t^d_{1-\kappa}$ and $t^d_{\alpha/2}$ should be drawn from an inverse t-distribution with the same degrees of freedom as the *ex post* regression model. This means that if researchers plan to use CRVE standard errors clustered by experimental unit, they should draw these critical values from an inverse t-distribution with J degrees of freedom. To be precise, these critical values should be sensitive to changes in the number of unit/clusters J, although the t degrees of freedom has a very small effect on MDE, relative to other parameters in Equation (5).

Finally, in panel RCTs with CRVE standard errors clustered by unit, the proportion of units treated P cannot be too large or too small. Our simulations have demonstrated that the Equation (5) performs poorly is $P < 0.1$ or $P > 0.9$, because the CRVE requires a sufficient number of clusters that are both treated and control.

This paper's accompanying software packages allow researchers to conduct analytical power calculations for difference-in-differences RCTs, via the functions `pc_dd_analytic` in Stata and `PCDDAnalytic` in `R`.[21] These programs implement the estimation version of the serial-correlation-robust power calculation formula, Equation (D2), allowing researchers to solve for effect size MDE, sample size J, or power κ, as functions of all other parameters. Users may provide assumed variance-covariance parameters, or alternatively our software will estimate these parameters directly by applying the above procedure to a dataset stored in memory.

D.2 Simulation-based power calculations

In cases where researchers have access to a representative pre-existing dataset, we recommend that they perform power calculations via simulation. This obviates the need to estimate *ex ante* variance-covariance parameters, and it ensures that *ex ante* power calculations assume the same experimental design, regression model, and variance estimator expected to be used in *ex post* analysis. Our accompanying software packages perform simulation-based power calculations, using the programs `pc_simulate` in Stata and `PCSimulate` in `R`. These programs implement the following algorithm:

1. Choose the following candidate parameters: sample size J, pre-treatment periods m, and post-treatment periods r, treatment ratio P, minimum detectable effect MDE, and significance level α. Let X_{it} denote the outcome variable of interest in the pre-existing dataset.

2. Randomly select J units from the representative dataset, and randomly select a range of $(m + r)$ consecutive time periods. This will serve as a simulated experimental dataset, with sample size J, m pre-treatment periods, and r post-treatment periods.

3. Randomly scramble a $[J \times 1]$ vector of PJ ones and $(1 - P)J$ zeros, rounding PJ to the nearest integer. Assign each of the J units to either treatment $(D = 1)$ or control $(D = 0)$, based on the order of this scrambled vector.

4. Construct an experimental outcome variable Y_{it}, where $Y_{it} = X_{it} + MDE$ for treated units in post-treatment periods, and $Y_{it} = X_{it}$ otherwise.

5. Using this simulated experimental dataset and the simulated outcome variable Y_{it}, implement the exact regression specification and variance estimator to be used in *ex post* analysis. Record

21. See Section D.4 for details on acquiring this software.

whether this model rejects the null hypotheses of zero treatment effects with significance level α (i.e. $H_0 : \tau = 0$).

6. Repeat Steps 2–5 many times, and calculate the rejection rate across all simulations. This is the experiment's statistical power as a function of J, m, r, P, MDE, and α.

7. Repeat Steps 1–6 for a range of MDEs and design parameters, increasing the number of simulations after narrowing down this range of parameters to more precisely calibrate power.

This algorithm allows users to test alternative regression specifications and alternative standard error assumptions, without needing to formally derive a power calculation expression for each model. If the pre-existing dataset contains fewer cross-sectional units than the desired sample size J, our software allows users to simulate additional units by bootstrapping units with replacement from the existing dataset (using the functions `pc_bootstrap_units` in Stata and `PCBootstrapUnits` in `R`). Unfortunately, if the pre-existing dataset contains fewer time periods than the desired panel length $(m+r)$, an analogous bootstrapping procedure would be much less straightforward (because unlike cross-sectional units, time periods are ordered and have a ordered covariance structure that is not orthogonal to the treatment vector D).

Importantly, this simulation-based algorithm can only calibrate statistical power κ. Rather than rely on the critical value $t^d_{1-\kappa}$, the algorithm simply estimates realized power as the proportion of simulations where the treatment effect is statistically distinguishable from zero. (By contrast, users may algebraically rearrange (or invert) an analytical power calculation formula to solve for any one of its parameters.) Calibrating simulation-based power calculations for a parameter other than κ necessitates a grid search over candidate parameter values, as described in Step 7 above. For example, to calibrate sample size J by simulation, users may repeat Step 1–6 over a range of candidate J values, narrowing this range (while simultaneously increasing the number of simulations) to calibrate to the desired power.

D.3 Lack of (representative) pre-existing data

To perform accurate *ex ante* power calculations, researchers must either have access to data that is representative (in expectation) of their future experimental data, or be able to parameterize an analytical power calculation with accurate estimates of the variance and covariance of the error structure. We recommend that researchers conduct power calculations via simulation (as described above), in cases where they have a representative pre-existing dataset with (i) data for the desired outcome (and relevant control variables); (ii) at least as many unique cross-sectional units as the desired experimental sample size; and (iii) a time series at least as long the desired experimental panel length. Many candidate experiments likely satisfy these criteria, such as when researchers partner with organizations that maintain historical databases on the desired population of experimental subjects.

At the same time, there are many cases where researchers cannot obtain representative data *ex ante*. This problem is not unique to panel data, as even the simple cross-sectional power calculation formula hinges on (an estimate of) the variance σ^2_ε. However, power calculations for panel RCT designs require four variance-covariance parameters: σ^2_ω, ψ^B, ψ^A, and ψ^X. While σ^2_ω is fixed in the population, the ψ (and $\psi_{\hat{\omega}}$) terms are endogenous to the panel length of the experiment, which underscores the importance of estimating $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$ from a representative time series.

In the absence of representative data, we generally recommend using analytical formulas in conjunction with appropriate sensitivity analyses.[22] Depending on the type of data that *is* available, approximating the parameters σ^2_ω, ψ^B, ψ^A, and ψ^X may be possible. We consider three cases:

1. *Too few units:* If researchers have access to a representative pre-existing dataset with too few cross-sectional units, they may still estimate $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$, and apply these values to the (estimation-specific) analytic formula.[23] These variance-covariance parameters do *not* depend on sample size J in the SCR power calculation formula, and estimates of $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$ derived from residuals are *not* sensitive in expectation to the number of panel units J to be used in the experiment.[24] Alternatively, we recommend that researchers bootstrap units by sampling existing units with replacement, and use this expanded dataset (including simulated units) to conduct power calculations by simulation, as described above.[25]

2. *Too few time periods:* If the pre-existing dataset contains too few time periods, researchers may still estimate $\tilde{\sigma}^2_{\hat{\omega}}$ using residuals from a regression with fewer than $m + r$ periods (because σ^2_ω does not depend on panel length). However, the ψ terms do depend on panel length, and they cannot be estimated directly from a dataset with fewer than $m+r$ periods. One strategy is to simply estimate $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$ using the longest possible panel (i.e., all available time periods in the pre-existing dataset), even if it is shorter than $m+r$ periods. The resulting $\psi_{\hat{\omega}}$ estimates are likely to be upper bounds (in absolute value) on the $\psi_{\hat{\omega}}$ estimates for longer panels, because as the panel length increases, the ψ terms incorporate more covariances between time periods that are further apart (which tend to become less correlated in distance). Another strategy is to attempt to extend the time series for each unit, analogous to the approach of bootstrapping units. As a rule of thumb, researchers often approximate time series data as an AR(k) process with $k \geq \sqrt[3]{T}$, where T is the full time series length. To extend short panels, researchers may estimate this AR(k) process using (residuals from) the existing dataset, and then simulate forward for each unit's outcome realization. Neither of these strategies is perfect, and we recommend conducting appropriate sensitivity analyses in either case.

3. *No data:* In the complete absence of data, power calculations will be challenging. At the very least, we recommend that researchers search for estimates of the residual variance in the existing literature, noting that panel fixed effects models are likely to yield lower residual variances than cross-sectional models with similar outcome data. If this is not possible, researchers may iterate analytical power calculations over a range of parameter choices. If researchers are able to guess a reasonable value of σ^2_ω, they my test a range of AR(1) parameters for plausible values of $\psi^{B,A,X}$. As a rule of thumb, $\psi^{B,A,X}$ are likely to be positive in the absence

22. The alternative would be to impose assumptions on some existing data to construct a simulated representative dataset, which could then be used to conduct power calculations by simulation. (This may involve structural assumptions on the data generating process and/or assumptions on the representativeness of the best available pre-existing dataset.) However, this process will be much more computationally intensive than simply applying an analytical formula with appropriate parameter sensitivities. The exception is the case where users simulate additional cross-sectional units by bootstrapping an existing dataset (Case 1 here), which facilitates simulation-based power calculations using both real and simulated experiment units.

23. It is important that researchers using *estimated* parameters use Equation (D2) rather than Equation (5).

24. Estimates of $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$ are sensitive to the number of cross-sectional units I used to estimate $\hat{\omega}_{it}$, but this is not related to the sample size parameter J. In Equation (D2), the comparative static $d\psi_{\hat{\omega}}/dJ = 0$.

25. Our accompanying software packages allow users to perform this bootstrapping algorithm using the functions `pc_bootstrap_units` in Stata and `PCBootstrapUnits` in `R`.

of a strong prior of negative serial correlation. In absolute value, $\psi^{B,A,X}$ should not exceed σ_ω^2, and they should decrease monotonically in panel length. To provide a sense of what plausible values of ψ^B, ψ^A, and ψ^X (and their residual-based counterparts) may be, we plot estimates from a range of panel lengths using simulated AR(1) data, the Bloom et al. (2015) data, and Pecan Street data, in Figures D1, D2, and D3, respectively.

D.4 Software packages

To facilitate user implementation of the methods described above, we are releasing accompanying software packages in Stata and in `R`. These packages each contain the following four programs:

`pc_simulate` (Stata), `PCSimulate` (`R`): Performs simulation-based power calculations using a pre-existing dataset. Accomodates four types of RCTs:

- **one-shot** (one wave of post-treatment data)
- **post-only** (multiple waves of post-treatment data)
- **difference-in-difference** (pre-treatment and post-treatment data)
- **ANCOVA** (post-treatment data, conditioning on pre-treatment data)

Calculates realized power over a set of user-provided design parameters. Allows for arbitrary linear regression specifications, linear controls, stratified randomization, and estimation of collapsed models.

`pc_bootstrap_units` (Stata), `PCBootstrapUnits` (`R`): Bootstraps additional units from an existing dataset, to allow for simulation-based power calculations user more cross-sectional units than are included in the pre-existing dataset.

`pc_dd_analytic` (Stata), `PCDDAnalytic` (`R`): Performs analytical power calculations using the serial-correlation-robust power calculation formula (Equation (5)). Solves for either sample size J, minimum detectable effect MDE, or power κ, as a function of all other parameter values. Users may either input assumed values for variance-covariance parameters, or allow the program to estimate variance-covariance parameters from data stored in memory.

`pc_dd_covar` (Stata), `PCDDCovar` (`R`): Estimates $\sigma_{\hat{\omega}}^2$, $\psi_{\hat{\omega}}^B$, $\psi_{\hat{\omega}}^A$, and $\psi_{\hat{\omega}}^X$ from an existing dataset, for a given number of pre-treatment and post-treatment periods. Serves as subprogram within `pc_dd_analytic`/`PCDDAnalytic`.

All of these programs are available for download. The Stata packages are available via `-ssc install pcpanel-`. The `R` packages remain in development, but will be posted shortly.

E Estimation-related proofs

In this section, we prove that researchers may calculate unbiased power calculations by estimating the variance-covariance parameters from a real dataset, where the parameters governing the data generating process is unknown.

E.1 Recovering estimated parameters

Here, we demonstrate that the procedure described in Appendix D.1 recovers unbiased estimates of the variance and covariance parameters governing the residuals $\hat{\omega}_{it}$ from a regression of Y_{it} on unit and time fixed effects, $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$ (though these do *not* represent unbiased estimates of the *true* parameters σ^2_{ω}, ψ^B, ψ^A, and ψ^X). We denote our procedure for computing these parameters with a $\sim$. Note that throughout this section, we are considering I units in the sample used to estimate $\hat{\omega}_{it}$, which may be distinct from the sample size J units used in the ensuing power calculations. Note also that because we are estimating the variance and covariance of a *population* of residuals, we use the population variance/covariance estimators as opposed to the (unbiased) sample variance/covariance estimators.[26]

In order to estimate the variance of the residuals, $\sigma^2_{\hat{\omega}}$, we simply calculate:

$$\begin{aligned}\tilde{\sigma}^2_{\hat{\omega}} &= \text{Var}\left(\hat{\omega}_{it} \mid \mathbf{X}\right) \\ &= \frac{1}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\left(\hat{\omega}_{it} - \bar{\bar{\hat{\omega}}}\right)^2\end{aligned}$$

where $\bar{\bar{\hat{\omega}}} = \frac{1}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\hat{\omega}_{it} = 0$. Taking expectations of both sides:

$$\begin{aligned}\text{E}\left[\tilde{\sigma}^2_{\hat{\omega}} \mid \mathbf{X}\right] &= \frac{1}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\text{E}\left[\hat{\omega}^2_{it} \mid \mathbf{X}\right] \\ &= \sigma^2_{\hat{\omega}}\end{aligned}$$

To estimate $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$, we define the $[I \times 1]$ vector of residuals for period t as $\vec{\hat{\omega}}_t$. This allows us to calculate:

$$\begin{aligned}\tilde{\psi}^B_{\hat{\omega}} &= \frac{2}{m(m-1)}\sum_{t=-m+1}^{-1}\sum_{s=t+1}^{0}\text{Cov}\left(\vec{\hat{\omega}}_t, \vec{\hat{\omega}}_s \mid \mathbf{X}\right) \\ &= \frac{2}{Im(m-1)}\sum_{t=-m+1}^{-1}\sum_{s=t+1}^{0}\sum_{i=1}^{I}\left(\hat{\omega}_{it}\hat{\omega}_{is} - \hat{\omega}_{it}\bar{\hat{\omega}}_s - \hat{\omega}_{is}\bar{\hat{\omega}}_t + \bar{\hat{\omega}}_t\bar{\hat{\omega}}_s\right)\end{aligned}$$

26. This means that to calculate $\tilde{\sigma}^2_{\hat{\omega}}$, we deflate the sample variance estimate by $\frac{IT-1}{IT}$, and to calculate the $\tilde{\psi}_{\hat{\omega}}$ terms, we deflate the sample covariance estimates by $\frac{I-1}{I}$. This distinction is ultimately innocuous, and the following derivations simply rely on a consistent decision to use either the population or sample variance/covariance estimators.

where $\bar{\hat{\omega}}_t = \frac{1}{I}\sum_{i=1}^{I}\hat{\omega}_{it} = 0$. Taking expectations yields:

$$\begin{aligned}
\mathrm{E}\left[\tilde{\psi}_{\hat{\omega}}^{B} \mid \mathbf{X}\right] &= \frac{2}{Im(m-1)}\sum_{t=-m+1}^{-1}\sum_{s=t+1}^{0}\sum_{i=1}^{I}\mathrm{E}\left[\hat{\omega}_{it}\hat{\omega}_{is} \mid \mathbf{X}\right] \\
&= \frac{2}{Im(m-1)}\sum_{t=-m+1}^{-1}\sum_{s=t+1}^{0}\sum_{i=1}^{I}\mathrm{Cov}(\hat{\omega}_{it}, \hat{\omega}_{is} \mid \mathbf{X}) \\
&= \psi_{\hat{\omega}}^{B}
\end{aligned}$$

Similarly:

$$\begin{aligned}
\tilde{\psi}_{\hat{\omega}}^{A} &= \frac{2}{r(r-1)}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}\left(\vec{\hat{\omega}}_t, \vec{\hat{\omega}}_s \mid \mathbf{X}\right) \\
&= \frac{2}{Ir(r-1)}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\sum_{i=1}^{I}\left(\hat{\omega}_{it}\hat{\omega}_{is} - \hat{\omega}_{it}\bar{\hat{\omega}}_s - \hat{\omega}_{is}\bar{\hat{\omega}}_t + \bar{\hat{\omega}}_t\bar{\hat{\omega}}_s\right)
\end{aligned}$$

and therefore:

$$\mathrm{E}\left[\tilde{\psi}_{\hat{\omega}}^{A} \mid \mathbf{X}\right] = \psi_{\hat{\omega}}^{A}$$

Applying the same steps to $\psi_{\hat{\omega}}^{X}$:

$$\begin{aligned}
\tilde{\psi}_{\hat{\omega}}^{X} &= \frac{1}{mr}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}\left(\vec{\hat{\omega}}_t, \vec{\hat{\omega}}_s \mid \mathbf{X}\right) \\
&= \frac{1}{Imr}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\sum_{i=1}^{I}\left(\hat{\omega}_{it}\hat{\omega}_{is} - \hat{\omega}_{it}\bar{\hat{\omega}}_s - \hat{\omega}_{is}\bar{\hat{\omega}}_t + \bar{\hat{\omega}}_t\bar{\hat{\omega}}_s\right)
\end{aligned}$$

Taking expectations of both sides:

$$\begin{aligned}
\mathrm{E}\left[\tilde{\psi}_{\hat{\omega}}^{X} \mid \mathbf{X}\right] &= \frac{1}{Imr}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\sum_{i=1}^{I}\mathrm{E}\left[\hat{\omega}_{it}\hat{\omega}_{is} \mid \mathbf{X}\right] \\
&= \frac{1}{Imr}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\sum_{i=1}^{I}\mathrm{Cov}\left(\hat{\omega}_{it}, \hat{\omega}_{is} \mid \mathbf{X}\right) \\
&= \psi_{\hat{\omega}}^{X}
\end{aligned}$$

Hence, we can recover unbiased estimates of the parameters $\sigma_{\hat{\omega}}^2$, $\psi_{\hat{\omega}}^{B}$, $\psi_{\hat{\omega}}^{A}$, and $\psi_{\hat{\omega}}^{X}$ (defined over residuals $\hat{\omega}_{it}$, rather than errors ω_{it}) by calculating the averages of the estimated $\tilde{\sigma}_{\hat{\omega}}^2$, $\tilde{\psi}_{\hat{\omega}}^{B}$, $\tilde{\psi}_{\hat{\omega}}^{A}$, and $\tilde{\psi}_{\hat{\omega}}^{X}$, respectively.

E.2 Estimating MDE from residual-based parameters

To calculate the MDE using the SCR formula, we must know the true parameters that characterize the variance and covariance of the error structure, σ^2_ω, ψ^B, ψ^A, and ψ^X. We cannot calculate these these parameters directly from a real dataset, however, because we do not observe the true error structure or data generating process. Instead, we estimate a residual for each observation and calculate the residual-based analogs of these parameters, $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$. In this section, we derive an expression for MDE^{est} in terms of these residual-based parameters that is equivalent to MDE from the SCR formula as defined in terms of true variance-covariance parameters:

$$MDE^{est}\left(\sigma^2_{\hat{\omega}}, \psi^B_{\hat{\omega}}, \psi^A_{\hat{\omega}}, \psi^X_{\hat{\omega}}\right) = MDE\left(\sigma^2_\omega, \psi^B, \psi^A, \psi^X\right)$$

Model While estimating the variance and covariance parameters of a dataset does not require a treatment, we assume that all other features of this model are identical to the model that generates the serial-correlation-robust power calculation formula, Equation (5).

That is, there are J units, P proportion of which are randomized into treatment. The researcher again collects outcome data Y_{it} for each unit i, across m pre-treatment time periods and r post-treatment time periods. For treated units, $D_{it} = 0$ in pre-treatment periods, and $D_{it} = 1$ in post-treatment periods; for control units, $D_{it} = 0$ in all periods. We restate the remaining assumptions from Section A.2.2 here for convenience:

Assumption (Data generating process). *The data are generated according to the following model:*

$$Y_{it} = \beta + \tau D_{it} + \upsilon_i + \delta_t + \omega_{it} \tag{E3}$$

where Y_{it} is the outcome of interest for unit i at time t; β is the expected outcome of non-treated observations; τ is the treatment effect that is homogenous across all units and all time periods; D_{it} is a time-varying treatment indicator; υ_i is a unit-specific disturbance distributed i.i.d. $\mathcal{N}(0, \sigma^2_\upsilon)$; δ_t is a time-specific disturbance distributed i.i.d. $\mathcal{N}(0, \sigma^2_\delta)$; and ω_{it} is an idiosyncratic error term distributed (not necessarily i.i.d.) $\mathcal{N}(0, \sigma^2_\omega)$.

Assumption (Strict exogeneity). $\mathrm{E}[\omega_{it} \mid \mathbf{X}] = 0$, *where $\mathbf{X}$ is a full rank matrix of regressors, including a constant, the treatment indicator D, $J-1$ unit fixed effects, and $(m+r)-1$ time fixed effects. This again follows from random assignment of D_{it}.*

Assumption (Balanced panel). *The number of pre-treatment observations, m, and post-treatment observations, r, is the same for each unit, and all units are observed in every time period.*

Assumption (Independence across units). $\mathrm{E}[\omega_{it}\omega_{js} \mid \mathbf{X}] = 0, \ \forall\, i \neq j, \ \forall\, t, s.$

Assumption (Symmetric covariance structures). *Define:*

$$\psi^B_T \equiv \frac{2}{PJm(m-1)} \sum_{i=1}^{PJ} \sum_{t=-m+1}^{-1} \sum_{s=t+1}^{0} \mathrm{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right)$$

$$\psi^A_T \equiv \frac{2}{PJr(r-1)} \sum_{i=1}^{PJ} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \mathrm{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right)$$

$$\psi_T^X \equiv \frac{1}{PJmr} \sum_{i=1}^{PJ} \sum_{t=-m+1}^{0} \sum_{s=1}^{r} \text{Cov}\left(\omega_{it}, \omega_{is} \mid \mathbf{X}\right)$$

to be the average pre-treatment, post-treatment, and across-period covariance between different error terms of the same treated unit, respectively. Define ψ_C^B, ψ_C^A, and ψ_C^X analogously, where we consider the $(1-P)J$ control units instead of the PJ treated units. Using these definitions, assume that $\psi^B = \psi_T^B = \psi_C^B$; $\psi^A = \psi_T^A = \psi_C^A$; and $\psi^X = \psi_T^X = \psi_C^X$.[27]

Estimates We first need to estimate the residuals of this model. To do this, we regress Y_{it} on a constant and fixed effects at the unit and time levels. For a balanced panel, the estimated coefficients are

$$\hat{\beta} = \frac{1}{IT} \sum_{i=1}^{I} \sum_{t=-m+1}^{r} Y_{it}$$

$$\hat{\upsilon}_i = \frac{1}{T} \sum_{t=-m+1}^{r} TY_{it} - \frac{1}{IT} \sum_{i=1}^{I} \sum_{t=-m+1}^{r} Y_{it}$$

$$\hat{\delta}_t = \frac{1}{I} \sum_{i=1}^{I} Y_{it} - \frac{1}{IT} \sum_{i=1}^{I} \sum_{t=-m+1}^{r} Y_{it}$$

Then the residual is defined as

$$\begin{aligned} \hat{\omega}_{it} &= Y_{it} - \hat{Y}_{it} \\ &= (\beta + \upsilon_i + \delta_t + \omega_{it}) - \left(\hat{\beta} + \hat{\upsilon}_i + \hat{\delta}_t\right) \\ &= \omega_{it} - \bar{\omega}_i - \bar{\omega}_t + \bar{\bar{\omega}} \end{aligned}$$

where

$$\bar{\omega}_i = \frac{1}{T} \sum_{s=-m+1}^{r} \omega_{is}$$

$$\bar{\omega}_t = \frac{1}{I} \sum_{j=1}^{I} \omega_{jt}$$

$$\bar{\bar{\omega}} = \frac{1}{IT} \sum_{j=1}^{I} \sum_{s=-m+1}^{r} \omega_{js}$$

27. We choose the letters "B" to indicate the Before-treatment period, and "A" to indicate the After-treatment period. We index the m pre-treatment periods $\{-m+1, \ldots, 0\}$, and the r post-treatment periods $\{1, \ldots, r\}$. In a randomized setting, $\text{E}[\psi_T^B] = \text{E}[\psi_C^B]$, $\text{E}[\psi_T^A] = \text{E}[\psi_C^A]$, and $\text{E}[\psi_T^X] = \text{E}[\psi_C^X]$, making this a reasonable assumption *ex ante*. However, it is possible for treatment to alter the covariance structure of treated units only.

We can now use this definition of residuals to derive expressions for $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$. We first derive an expression for $\sigma^2_{\hat{\omega}}$, the average variance of a residual.

$$
\begin{aligned}
\sigma^2_{\hat{\omega}} &= \frac{1}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r} \mathrm{Var}(\hat{\omega}_{it} \mid \mathbf{X}) \\
&= \frac{1}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r} \mathrm{Var}\left(\omega_{it} - \bar{\omega}_i - \bar{\omega}_t + \bar{\bar{\omega}} \mid \mathbf{X}\right) \\
&= \frac{1}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r} \left[\mathrm{Var}(\omega_{it} \mid \mathbf{X}) + \mathrm{Var}\left(\bar{\omega}_i \mid \mathbf{X}\right) + \mathrm{Var}\left(\bar{\omega}_t \mid \mathbf{X}\right) + \mathrm{Var}\left(\bar{\bar{\omega}} \mid \mathbf{X}\right)\right. \\
&\qquad - 2\,\mathrm{Cov}\left(\omega_{it}, \bar{\omega}_i \mid \mathbf{X}\right) - 2\,\mathrm{Cov}\left(\omega_{it}, \bar{\omega}_t \mid \mathbf{X}\right) + 2\,\mathrm{Cov}\left(\omega_{it}, \bar{\bar{\omega}} \mid \mathbf{X}\right) \\
&\qquad \left. +2\,\mathrm{Cov}\left(\bar{\omega}_i, \bar{\omega}_t \mid \mathbf{X}\right) - 2\,\mathrm{Cov}\left(\bar{\omega}_i, \bar{\bar{\omega}} \mid \mathbf{X}\right) - 2\,\mathrm{Cov}\left(\bar{\omega}_t, \bar{\bar{\omega}} \mid \mathbf{X}\right)\right]
\end{aligned}
$$

Calculating each of these terms, in turn, gives

$$
\frac{1}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r} \mathrm{Var}(\omega_{it} \mid \mathbf{X}) = \sigma^2_{\omega}
$$

$$
\begin{aligned}
\frac{1}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r} \mathrm{Var}\left(\bar{\omega}_i \mid \mathbf{X}\right) &= \frac{1}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r} \mathrm{Var}\left(\frac{1}{T}\sum_{s=-m+1}^{r} \omega_{is} \mid \mathbf{X}\right) \\
&= \frac{1}{IT^3}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\sum_{s=-m+1}^{r} \mathrm{Var}\left(\omega_{is} \mid \mathbf{X}\right) \\
&\quad + \frac{2}{IT^3}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=-m+1}^{r-1}\sum_{p=s+1}^{r} \mathrm{Cov}(\omega_{is}, \omega_{ip} \mid \mathbf{X}) \\
&= \frac{1}{IT^2}\sum_{i=1}^{I}\sum_{s=-m+1}^{r} \mathrm{Var}\left(\omega_{is} \mid \mathbf{X}\right) \\
&\quad + \frac{2}{IT^2}\left[\sum_{i=1}^{I}\sum_{s=-m+1}^{-1}\sum_{p=s+1}^{0} \mathrm{Cov}(\omega_{is}, \omega_{ip} \mid \mathbf{X})\right. \\
&\qquad + \sum_{i=1}^{I}\sum_{s=-m+1}^{0}\sum_{p=1}^{r} \mathrm{Cov}(\omega_{is}, \omega_{ip} \mid \mathbf{X}) \\
&\qquad \left. + \sum_{i=1}^{I}\sum_{s=1}^{r-1}\sum_{p=s+1}^{r} \mathrm{Cov}(\omega_{is}, \omega_{ip} \mid \mathbf{X})\right] \\
&= \frac{1}{T}\sigma^2_{\omega} + \frac{m(m-1)}{T^2}\psi^B + \frac{r(r-1)}{T^2}\psi^A + \frac{2mr}{T^2}\psi^X
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r} \mathrm{Var}\left(\bar{\omega}_t \mid \mathbf{X}\right) &= \frac{1}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r} \mathrm{Var}\left(\frac{1}{I}\sum_{j=1}^{I} \omega_{jt} \mid \mathbf{X}\right) \\
&= \frac{1}{I^3T}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=-m+1}^{r} \mathrm{Var}\left(\omega_{jt} \mid \mathbf{X}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{2}{I^3T}\sum_{i=1}^{I}\sum_{j=1}^{I-1}\sum_{k=j+1}^{I}\sum_{t=-m+1}^{0}\mathrm{Cov}(\omega_{jt},\omega_{kt}\mid \mathbf{X})\\
&= \frac{1}{I^2T}\sum_{j=1}^{I}\sum_{t=-m+1}^{r}\mathrm{Var}\left(\omega_{jt}\mid \mathbf{X}\right)\\
&= \frac{1}{I}\sigma_\omega^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\mathrm{Var}\left(\bar{\bar{\omega}}\mid \mathbf{X}\right) &= \frac{1}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\mathrm{Var}\left(\frac{1}{IT}\sum_{j=1}^{I}\sum_{s=-m+1}^{r}\omega_{js}\mid \mathbf{X}\right)\\
&= \frac{1}{I^3T^3}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=-m+1}^{r}\sum_{s=-m+1}^{r}\mathrm{Var}(\omega_{js}\mid \mathbf{X})\\
&+ \frac{2}{I^3T^3}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=-m+1}^{r}\sum_{s=-m+1}^{r-1}\sum_{p=s+1}^{r}\mathrm{Cov}(\omega_{js},\omega_{jp}\mid \mathbf{X})\\
&+ \frac{2}{I^3T^3}\sum_{i=1}^{I}\sum_{j=1}^{I-1}\sum_{k=j+1}^{I}\sum_{t=-m+1}^{r}\sum_{s=-m+1}^{r}\sum_{p=-m+1}^{r}\mathrm{Cov}(\omega_{js},\omega_{kp}\mid \mathbf{X})\\
&= \frac{1}{I^2T^2}\sum_{j=1}^{I}\sum_{s=-m+1}^{r}\mathrm{Var}(\omega_{js}\mid \mathbf{X})\\
&+ \frac{2}{I^2T^2}\left[\sum_{j=1}^{I}\sum_{s=-m+1}^{-1}\sum_{p=s+1}^{0}\mathrm{Cov}(\omega_{js},\omega_{jp}\mid \mathbf{X})\right.\\
&\qquad + \sum_{j=1}^{I}\sum_{s=-m+1}^{0}\sum_{p=1}^{r}\mathrm{Cov}(\omega_{js},\omega_{jp}\mid \mathbf{X})\\
&\qquad \left. + \sum_{j=1}^{I}\sum_{s=1}^{r-1}\sum_{p=s+1}^{r}\mathrm{Cov}(\omega_{js},\omega_{jp}\mid \mathbf{X})\right]\\
&= \frac{1}{IT}\sigma_\omega^2 + \frac{m(m-1)}{IT^2}\psi^B + \frac{r(r-1)}{IT^2}\psi^A + \frac{2mr}{IT^2}\psi^X
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}2\,\mathrm{Cov}\left(\omega_{it},\bar{\omega}_i\mid \mathbf{X}\right) &= \frac{2}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\mathrm{Cov}\left(\omega_{it},\frac{1}{T}\sum_{s=-m+1}^{r}\omega_{is}\mid \mathbf{X}\right)\\
&= \frac{2}{IT^2}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\sum_{s=-m+1}^{r}\mathrm{Cov}(\omega_{it},\omega_{is}\mid \mathbf{X})\\
&= \frac{2}{IT^2}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\mathrm{Var}(\omega_{it}\mid \mathbf{X})\\
&+ \frac{4}{IT^2}\sum_{i=1}^{I}\sum_{t=-m+1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}(\omega_{it},\omega_{is}\mid \mathbf{X})\\
&= \frac{2}{IT^2}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\mathrm{Var}(\omega_{it}\mid \mathbf{X})
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{4}{IT^2}\Bigg[\sum_{i=1}^{I}\sum_{t=-m+1}^{-1}\sum_{s=t+1}^{0}\operatorname{Cov}(\omega_{it},\omega_{is}\mid \mathbf{X}) \\
&\qquad + \sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\operatorname{Cov}(\omega_{it},\omega_{is}\mid \mathbf{X}) \\
&\qquad + \sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\operatorname{Cov}(\omega_{it},\omega_{is}\mid \mathbf{X})\Bigg] \\
&= \frac{2}{T}\sigma_\omega^2 + \frac{2m(m-1)}{T^2}\psi^B + \frac{2r(r-1)}{T^2}\psi^A + \frac{4mr}{T^2}\psi^X
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r} 2\operatorname{Cov}(\omega_{it},\bar{\omega}_t\mid \mathbf{X}) &= \frac{2}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\operatorname{Cov}\left(\omega_{it},\frac{1}{I}\sum_{j=1}^{I}\omega_{jt}\mid \mathbf{X}\right) \\
&= \frac{2}{I^2T}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=-m+1}^{r}\operatorname{Cov}(\omega_{it},\omega_{jt}\mid \mathbf{X}) \\
&= \frac{2}{I^2T}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\operatorname{Var}(\omega_{it}\mid \mathbf{X}) \\
&= \frac{2}{I}\sigma_\omega^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r} 2\operatorname{Cov}(\omega_{it},\bar{\bar{\omega}}\mid \mathbf{X}) &= \frac{2}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\operatorname{Cov}\left(\omega_{it},\frac{1}{IT}\sum_{j=1}^{I}\sum_{s=-m+1}^{r}\omega_{js}\mid \mathbf{X}\right) \\
&= \frac{2}{I^2T^2}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=-m+1}^{r}\sum_{s=-m+1}^{r}\operatorname{Cov}(\omega_{it},\omega_{js}\mid \mathbf{X}) \\
&= \frac{2}{I^2T^2}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\operatorname{Var}(\omega_{it}\mid \mathbf{X}) \\
&\quad + \frac{4}{I^2T^2}\sum_{i=1}^{I}\sum_{t=-m+1}^{r-1}\sum_{s=t+1}^{r}\operatorname{Cov}(\omega_{it},\omega_{is}\mid \mathbf{X}) \\
&= \frac{2}{I^2T^2}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\operatorname{Var}(\omega_{it}\mid \mathbf{X}) \\
&\quad + \frac{4}{I^2T^2}\Bigg[\sum_{i=1}^{I}\sum_{t=-m+1}^{-1}\sum_{s=t+1}^{0}\operatorname{Cov}(\omega_{it},\omega_{is}\mid \mathbf{X}) \\
&\qquad + \sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\operatorname{Cov}(\omega_{it},\omega_{is}\mid \mathbf{X}) \\
&\qquad + \sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\operatorname{Cov}(\omega_{it},\omega_{is}\mid \mathbf{X})\Bigg] \\
&= \frac{2}{IT}\sigma_\omega^2 + \frac{2m(m-1)}{IT^2}\psi^B + \frac{2r(r-1)}{IT^2}\psi^A + \frac{4mr}{IT^2}\psi^X
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r} 2\operatorname{Cov}\left(\bar{\omega}_i,\bar{\omega}_t \mid \mathbf{X}\right) &= \frac{2}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\operatorname{Cov}\left(\frac{1}{T}\sum_{s=-m+1}^{r}\omega_{is},\frac{1}{I}\sum_{j=1}^{I}\omega_{jt} \mid \mathbf{X}\right)\\
&= \frac{2}{I^2T^2}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=-m+1}^{r}\sum_{s=-m+1}^{r}\operatorname{Cov}(\omega_{is},\omega_{jt} \mid \mathbf{X})\\
&= \frac{2}{I^2T^2}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\operatorname{Var}(\omega_{it} \mid \mathbf{X})\\
&\quad + \frac{4}{I^2T^2}\sum_{i=1}^{I}\sum_{t=-m+1}^{r-1}\sum_{s=t+1}^{r}\operatorname{Cov}(\omega_{it},\omega_{is} \mid \mathbf{X})\\
&= \frac{2}{I^2T^2}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\operatorname{Var}(\omega_{it} \mid \mathbf{X})\\
&\quad + \frac{4}{I^2T^2}\left[\sum_{i=1}^{I}\sum_{t=-m+1}^{-1}\sum_{s=t+1}^{0}\operatorname{Cov}(\omega_{it},\omega_{is} \mid \mathbf{X})\right.\\
&\qquad + \sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\operatorname{Cov}(\omega_{it},\omega_{is} \mid \mathbf{X})\\
&\qquad \left.+ \sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\operatorname{Cov}(\omega_{it},\omega_{is} \mid \mathbf{X})\right]\\
&= \frac{2}{IT}\sigma_\omega^2 + \frac{2m(m-1)}{IT^2}\psi^B + \frac{2r(r-1)}{IT^2}\psi^A + \frac{4mr}{IT^2}\psi^X
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r} 2\operatorname{Cov}\left(\bar{\omega}_i,\bar{\bar{\omega}} \mid \mathbf{X}\right) &= \frac{2}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\operatorname{Cov}\left(\frac{1}{T}\sum_{s=-m+1}^{r}\omega_{is},\frac{1}{IT}\sum_{j=1}^{I}\sum_{p=-m+1}^{r}\omega_{jp} \mid \mathbf{X}\right)\\
&= \frac{2}{I^2T^3}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=-m+1}^{r}\sum_{s=-m+1}^{r}\sum_{p=-m+1}^{r}\operatorname{Cov}(\omega_{is},\omega_{jp} \mid \mathbf{X})\\
&= \frac{2}{I^2T^3}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\sum_{s=-m+1}^{r}\operatorname{Var}(\omega_{is} \mid \mathbf{X})\\
&\quad + \frac{4}{I^2T^3}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\sum_{s=-m+1}^{r-1}\sum_{p=s+1}^{r}\operatorname{Cov}(\omega_{is},\omega_{ip} \mid \mathbf{X})\\
&= \frac{2}{I^2T^2}\sum_{i=1}^{I}\sum_{s=-m+1}^{r}\operatorname{Var}(\omega_{is} \mid \mathbf{X})\\
&\quad + \frac{4}{I^2T^2}\left[\sum_{i=1}^{I}\sum_{s=-m+1}^{-1}\sum_{p=s+1}^{0}\operatorname{Cov}(\omega_{is},\omega_{ip} \mid \mathbf{X})\right.\\
&\qquad + \sum_{i=1}^{I}\sum_{s=-m+1}^{0}\sum_{p=1}^{r}\operatorname{Cov}(\omega_{is},\omega_{ip} \mid \mathbf{X})
\end{aligned}
$$

$$+ \sum_{i=1}^{I}\sum_{s=1}^{r-1}\sum_{p=s+1}^{r} \mathrm{Cov}(\omega_{is}, \omega_{ip} \mid \mathbf{X})\Bigg]$$

$$= \frac{2}{IT}\sigma_\omega^2 + \frac{2m(m-1)}{IT^2}\psi^B + \frac{2r(r-1)}{IT^2}\psi^A + \frac{4mr}{IT^2}\psi^X$$

$$\frac{1}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r} 2\,\mathrm{Cov}\left(\bar{\omega}_t, \bar{\bar{\omega}} \mid \mathbf{X}\right) = \frac{2}{IT}\sum_{i=1}^{I}\sum_{t=-m+1}^{r}\mathrm{Cov}\left(\frac{1}{I}\sum_{j=1}^{I}\omega_{jt}, \frac{1}{IT}\sum_{k=1}^{I}\sum_{s=-m+1}^{r}\omega_{ks} \mid \mathbf{X}\right)$$

$$= \frac{2}{I^3T^2}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{k=1}^{I}\sum_{t=-m+1}^{r}\sum_{s=-m+1}^{r}\mathrm{Cov}(\omega_{jt}, \omega_{ks} \mid \mathbf{X})$$

$$= \frac{2}{I^3T^2}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=-m+1}^{r}\mathrm{Var}(\omega_{jt} \mid \mathbf{X})$$

$$+ \frac{4}{I^3T^2}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=-m+1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}(\omega_{jt}, \omega_{js} \mid \mathbf{X})$$

$$= \frac{2}{I^2T^2}\sum_{j=1}^{I}\sum_{t=-m+1}^{r}\mathrm{Var}(\omega_{jt} \mid \mathbf{X})$$

$$+ \frac{4}{I^2T^2}\Bigg[\sum_{j=1}^{I}\sum_{t=-m+1}^{-1}\sum_{s=t+1}^{0}\mathrm{Cov}(\omega_{jt}, \omega_{js} \mid \mathbf{X})$$

$$+ \sum_{j=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}(\omega_{jt}, \omega_{js} \mid \mathbf{X})$$

$$+ \sum_{j=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}(\omega_{jt}, \omega_{js} \mid \mathbf{X})\Bigg]$$

$$= \frac{2}{IT}\sigma_\omega^2 + \frac{2m(m-1)}{IT^2}\psi^B + \frac{2r(r-1)}{IT^2}\psi^A + \frac{4mr}{IT^2}\psi^X$$

Combining these terms yields

$$\sigma_{\hat{\omega}}^2 = \sigma_\omega^2 + \left[\frac{1}{T}\sigma_\omega^2 + \frac{m(m-1)}{T^2}\psi^B + \frac{r(r-1)}{T^2}\psi^A + \frac{2mr}{T^2}\psi^X\right] + \frac{1}{I}\sigma_\omega^2$$
$$+ \left[\frac{1}{IT}\sigma_\omega^2 + \frac{m(m-1)}{IT^2}\psi^B + \frac{r(r-1)}{IT^2}\psi^A + \frac{2mr}{IT^2}\psi^X\right]$$
$$- \left[\frac{2}{T}\sigma_\omega^2 + \frac{2m(m-1)}{T^2}\psi^B + \frac{2r(r-1)}{T^2}\psi^A + \frac{4mr}{T^2}\psi^X\right] - \frac{2}{I}\sigma_\omega^2$$
$$+ \left[\frac{2}{IT}\sigma_\omega^2 + \frac{2m(m-1)}{IT^2}\psi^B + \frac{2r(r-1)}{IT^2}\psi^A + \frac{4mr}{IT^2}\psi^X\right]$$
$$+ \left[\frac{2}{IT}\sigma_\omega^2 + \frac{2m(m-1)}{IT^2}\psi^B + \frac{2r(r-1)}{IT^2}\psi^A + \frac{4mr}{IT^2}\psi^X\right]$$
$$- \left[\frac{2}{IT}\sigma_\omega^2 + \frac{2m(m-1)}{IT^2}\psi^B + \frac{2r(r-1)}{IT^2}\psi^A + \frac{4mr}{IT^2}\psi^X\right]$$

$$
\begin{aligned}
&- \left[\frac{2}{IT}\sigma_\omega^2 + \frac{2m(m-1)}{IT^2}\psi^B + \frac{2r(r-1)}{IT^2}\psi^A + \frac{4mr}{IT^2}\psi^X\right] \\
&= \sigma_\omega^2 + \left[\frac{1}{T}\sigma_\omega^2 + \frac{m(m-1)}{T^2}\psi^B + \frac{r(r-1)}{T^2}\psi^A + \frac{2mr}{T^2}\psi^X\right] + \frac{1}{I}\sigma_\omega^2 \\
&\quad + \left[\frac{1}{IT}\sigma_\omega^2 + \frac{m(m-1)}{IT^2}\psi^B + \frac{r(r-1)}{IT^2}\psi^A + \frac{2mr}{IT^2}\psi^X\right] \\
&\quad - \left[\frac{2}{T}\sigma_\omega^2 + \frac{2m(m-1)}{T^2}\psi^B + \frac{2r(r-1)}{T^2}\psi^A + \frac{4mr}{T^2}\psi^X\right] - \frac{2}{I}\sigma_\omega^2 \\
&= \left[1 + \frac{1}{T} + \frac{1}{I} + \frac{1}{IT} - \frac{2}{T}\right]\sigma_\omega^2 + \left[\frac{m(m-1)}{T^2} + \frac{m(m-1)}{IT^2} - \frac{2m(m-1)}{T^2}\right]\psi^B \\
&\quad + \left[\frac{r(r-1)}{T^2} + \frac{r(r-1)}{IT^2} - \frac{2r(r-1)}{T^2}\right]\psi^A + \left[\frac{2mr}{T^2} + \frac{2mr}{IT^2} - \frac{4mr}{T^2}\right]\psi^X \\
&= \left(\frac{(I-1)(T-1)}{IT}\right)\sigma_\omega^2 - \left(\frac{(I-1)m(m-1)}{IT^2}\right)\psi^B - \left(\frac{(I-1)r(r-1)}{IT^2}\right)\psi^A - \left(\frac{2(I-1)mr}{IT^2}\right)\psi^X
\end{aligned}
$$

We next derive an expression for $\psi^A_{\hat{\omega}}$.

$$
\begin{aligned}
\psi^A_{\hat{\omega}} &= \frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r} \operatorname{Cov}(\hat{\omega}_{it}, \hat{\omega}_{is} \mid \mathbf{X}) \\
&= \frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r} \operatorname{Cov}(\omega_{it} - \bar{\omega}_i - \bar{\omega}_t + \bar{\bar{\omega}}, \omega_{is} - \bar{\omega}_i - \bar{\omega}_s + \bar{\bar{\omega}} \mid \mathbf{X}) \\
&= \frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r} [\operatorname{Cov}(\omega_{it}, \omega_{is} \mid \mathbf{X}) - \operatorname{Cov}(\omega_{it}, \bar{\omega}_i \mid \mathbf{X}) - \operatorname{Cov}(\omega_{it}, \bar{\omega}_s \mid \mathbf{X}) + \operatorname{Cov}(\omega_{it}, \bar{\bar{\omega}} \mid \mathbf{X}) \\
&\qquad - \operatorname{Cov}(\bar{\omega}_i, \omega_{is} \mid \mathbf{X}) + \operatorname{Cov}(\bar{\omega}_i, \bar{\omega}_i \mid \mathbf{X}) + \operatorname{Cov}(\bar{\omega}_i, \bar{\omega}_s \mid \mathbf{X}) - \operatorname{Cov}(\bar{\omega}_i, \bar{\bar{\omega}} \mid \mathbf{X}) \\
&\qquad - \operatorname{Cov}(\bar{\omega}_t, \omega_{is} \mid \mathbf{X}) + \operatorname{Cov}(\bar{\omega}_t, \bar{\omega}_i \mid \mathbf{X}) + \operatorname{Cov}(\bar{\omega}_t, \bar{\omega}_s \mid \mathbf{X}) - \operatorname{Cov}(\bar{\omega}_t, \bar{\bar{\omega}} \mid \mathbf{X}) \\
&\qquad + \operatorname{Cov}(\bar{\bar{\omega}}, \omega_{is} \mid \mathbf{X}) - \operatorname{Cov}(\bar{\bar{\omega}}, \bar{\omega}_i \mid \mathbf{X}) - \operatorname{Cov}(\bar{\bar{\omega}}, \bar{\omega}_s \mid \mathbf{X}) + \operatorname{Cov}(\bar{\bar{\omega}}, \bar{\bar{\omega}} \mid \mathbf{X})]
\end{aligned}
$$

We again calculate each term.

$$
\frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r} \operatorname{Cov}(\omega_{it}, \omega_{is} \mid \mathbf{X}) = \psi^A
$$

$$
\begin{aligned}
\frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r} \operatorname{Cov}(\omega_{it}, \bar{\omega}_i \mid \mathbf{X}) &= \frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r} \operatorname{Cov}\left(\omega_{it}, \frac{1}{T}\sum_{p=-m+1}^{r}\omega_{ip} \mid \mathbf{X}\right) \\
&= \frac{2}{r(r-1)IT}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\sum_{p=-m+1}^{r} \operatorname{Cov}(\omega_{it}, \omega_{ip} \mid \mathbf{X}) \\
&= \frac{2}{r(r-1)IT}\sum_{i=1}^{I}\sum_{t=1}^{r}\sum_{p=-m+1}^{r} (r-t)\operatorname{Cov}(\omega_{it}, \omega_{ip} \mid \mathbf{X})
\end{aligned}
$$

$$
\begin{aligned}
\frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}(\omega_{it},\bar{\omega}_s \mid \mathbf{X}) &= \frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}\left(\omega_{it},\frac{1}{I}\sum_{j=1}^{I}\omega_{js} \mid \mathbf{X}\right)\\
&= \frac{2}{r(r-1)I^2}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}(\omega_{it},\omega_{js} \mid \mathbf{X})\\
&= \frac{2}{r(r-1)I^2}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}(\omega_{it},\omega_{is} \mid \mathbf{X})\\
&= \frac{1}{I}\psi^A\\
\frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}(\omega_{it},\bar{\bar{\omega}} \mid \mathbf{X}) &= \frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}\left(\omega_{it},\frac{1}{IT}\sum_{j=1}^{I}\sum_{p=-m+1}^{r}\omega_{jp} \mid \mathbf{X}\right)\\
&= \frac{2}{r(r-1)I^2T}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\sum_{p=-m+1}^{r}\mathrm{Cov}(\omega_{it},\omega_{jp} \mid \mathbf{X})\\
&= \frac{2}{r(r-1)I^2T}\sum_{i=1}^{I}\sum_{t=1}^{r}\sum_{p=-m+1}^{r}(r-t)\,\mathrm{Cov}(\omega_{it},\omega_{ip} \mid \mathbf{X})\\
\frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}(\bar{\omega}_i,\omega_{is} \mid \mathbf{X}) &= \frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}\left(\frac{1}{T}\sum_{p=-m+1}^{r}\omega_{ip},\omega_{is} \mid \mathbf{X}\right)\\
&= \frac{2}{r(r-1)IT}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\sum_{p=-m+1}^{r}\mathrm{Cov}(\omega_{ip},\omega_{is} \mid \mathbf{X})\\
&= \frac{2}{r(r-1)IT}\sum_{i=1}^{I}\sum_{t=1}^{r}\sum_{p=-m+1}^{r}(t-1)\,\mathrm{Cov}\left(\omega_{it},\omega_{ip} \mid \mathbf{X}\right)\\
\frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}(\bar{\omega}_i,\bar{\omega}_i \mid \mathbf{X}) &= \frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}\left(\frac{1}{T}\sum_{p=-m+1}^{r}\omega_{ip},\frac{1}{T}\sum_{q=-m+1}^{r}\omega_{iq} \mid \mathbf{X}\right)\\
&= \frac{2}{r(r-1)IT^2}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\sum_{p=-m+1}^{r}\sum_{q=-m+1}^{r}\mathrm{Cov}\left(\omega_{ip},\omega_{iq} \mid \mathbf{X}\right)\\
&= \frac{2}{r(r-1)IT^2}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\sum_{p=-m+1}^{r}\mathrm{Var}\left(\omega_{ip} \mid \mathbf{X}\right)\\
&\quad+\frac{4}{r(r-1)IT^2}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\sum_{p=-m+1}^{r-1}\sum_{q=p+1}^{r}\mathrm{Cov}\left(\omega_{ip},\omega_{iq} \mid \mathbf{X}\right)\\
&= \frac{1}{IT^2}\sum_{i=1}^{I}\sum_{p=-m+1}^{r}\mathrm{Var}\left(\omega_{ip} \mid \mathbf{X}\right)\\
&\quad+\frac{2}{IT^2}\left[\sum_{i=1}^{I}\sum_{p=-m+1}^{-1}\sum_{q=p+1}^{0}\mathrm{Cov}\left(\omega_{ip},\omega_{iq} \mid \mathbf{X}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{I}\sum_{p=-m+1}^{0}\sum_{q=1}^{r}\operatorname{Cov}\left(\omega_{ip},\omega_{iq}\mid\mathbf{X}\right)\\
&\left.+\sum_{i=1}^{I}\sum_{p=1}^{r-1}\sum_{q=p+1}^{r}\operatorname{Cov}\left(\omega_{ip},\omega_{iq}\mid\mathbf{X}\right)\right]\\
&=\frac{1}{T}\sigma_\omega^2+\frac{m(m-1)}{T^2}\psi^B+\frac{r(r-1)}{T^2}\psi^A+\frac{2mr}{T^2}\psi^X
\end{aligned}
$$

$$
\begin{aligned}
\frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\operatorname{Cov}(\bar{\omega}_i,\bar{\omega}_s\mid\mathbf{X})&=\frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\operatorname{Cov}\left(\frac{1}{T}\sum_{p=-m+1}^{r}\omega_{ip},\frac{1}{I}\sum_{j=1}^{I}\omega_{js}\mid\mathbf{X}\right)\\
&=\frac{2}{r(r-1)I^2T}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\sum_{p=-m+1}^{r}\operatorname{Cov}(\omega_{ip},\omega_{js}\mid\mathbf{X})\\
&=\frac{2}{r(r-1)I^2T}\sum_{i=1}^{I}\sum_{t=1}^{r}\sum_{p=-m+1}^{r}(t-1)\operatorname{Cov}(\omega_{it},\omega_{ip}\mid\mathbf{X})
\end{aligned}
$$

$$
\begin{aligned}
\frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\operatorname{Cov}(\bar{\omega}_i,\bar{\bar{\omega}}\mid\mathbf{X})&=\frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\operatorname{Cov}\left(\frac{1}{T}\sum_{p=-m+1}^{r}\omega_{ip},\frac{1}{IT}\sum_{j=1}^{I}\sum_{q=-m+1}^{r}\omega_{jq}\mid\mathbf{X}\right)\\
&=\frac{2}{r(r-1)I^2T^2}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\sum_{p=-m+1}^{r}\sum_{q=-m+1}^{r}\operatorname{Cov}(\omega_{ip},\omega_{jq}\mid\mathbf{X})\\
&=\frac{2}{r(r-1)I^2T^2}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\sum_{p=-m+1}^{r}\operatorname{Var}(\omega_{ip}\mid\mathbf{X})\\
&\quad+\frac{4}{r(r-1)I^2T^2}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\sum_{p=-m+1}^{r-1}\sum_{q=p+1}^{r}\operatorname{Cov}(\omega_{ip},\omega_{iq}\mid\mathbf{X})\\
&=\frac{1}{I^2T^2}\sum_{i=1}^{I}\sum_{p=-m+1}^{r}\operatorname{Var}(\omega_{ip}\mid\mathbf{X})\\
&\quad+\frac{2}{I^2T^2}\left[\sum_{i=1}^{I}\sum_{p=-m+1}^{-1}\sum_{q=p+1}^{0}\operatorname{Cov}(\omega_{ip},\omega_{iq}\mid\mathbf{X})\right.\\
&\qquad+\sum_{i=1}^{I}\sum_{p=-m+1}^{0}\sum_{q=1}^{r}\operatorname{Cov}(\omega_{ip},\omega_{iq}\mid\mathbf{X})\\
&\qquad\left.+\sum_{i=1}^{I}\sum_{p=1}^{r-1}\sum_{q=p+1}^{r}\operatorname{Cov}(\omega_{ip},\omega_{iq}\mid\mathbf{X})\right]\\
&=\frac{1}{IT}\sigma_\omega^2+\frac{m(m-1)}{IT^2}\psi^B+\frac{r(r-1)}{IT^2}\psi^A+\frac{2mr}{IT^2}\psi^X
\end{aligned}
$$

$$
\begin{aligned}
\frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\operatorname{Cov}(\bar{\omega}_t,\omega_{is}\mid\mathbf{X})&=\frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\operatorname{Cov}\left(\frac{1}{I}\sum_{j=1}^{I}\omega_{jt},\omega_{is}\mid\mathbf{X}\right)\\
&=\frac{2}{r(r-1)I^2}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\operatorname{Cov}(\omega_{jt},\omega_{is}\mid\mathbf{X})
\end{aligned}
$$

$$= \frac{2}{r(r-1)I^2} \sum_{i=1}^{I} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \operatorname{Cov}(\omega_{it}, \omega_{is} \mid \mathbf{X})$$

$$= \frac{1}{I} \psi^A$$

$$\frac{2}{r(r-1)I} \sum_{i=1}^{I} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \operatorname{Cov}(\bar{\omega}_t, \bar{\omega}_i \mid \mathbf{X}) = \frac{2}{r(r-1)I} \sum_{i=1}^{I} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \operatorname{Cov}\left(\frac{1}{I} \sum_{j=1}^{I} \omega_{jt}, \frac{1}{T} \sum_{p=-m+1}^{r} \omega_{ip} \mid \mathbf{X} \right)$$

$$= \frac{2}{r(r-1)I^2 T} \sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \sum_{p=-m+1}^{r} \operatorname{Cov}(\omega_{jt}, \omega_{ip} \mid \mathbf{X})$$

$$= \frac{2}{r(r-1)I^2 T} \sum_{i=1}^{I} \sum_{t=1}^{r} \sum_{p=-m+1}^{r} (r-t) \operatorname{Cov}(\omega_{it}, \omega_{ip} \mid \mathbf{X})$$

$$\frac{2}{r(r-1)I} \sum_{i=1}^{I} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \operatorname{Cov}(\bar{\omega}_t, \bar{\omega}_s \mid \mathbf{X}) = \frac{2}{r(r-1)I} \sum_{i=1}^{I} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \operatorname{Cov}\left(\frac{1}{I} \sum_{j=1}^{I} \omega_{jt}, \frac{1}{I} \sum_{k=1}^{I} \omega_{ks} \mid \mathbf{X} \right)$$

$$= \frac{2}{r(r-1)I^3} \sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{k=1}^{I} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \operatorname{Cov}(\omega_{jt}, \omega_{ks} \mid \mathbf{X})$$

$$= \frac{2}{r(r-1)I^2} \sum_{j=1}^{I} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \operatorname{Cov}(\omega_{jt}, \omega_{js} \mid \mathbf{X})$$

$$= \frac{1}{I} \psi^A$$

$$\frac{2}{r(r-1)I} \sum_{i=1}^{I} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \operatorname{Cov}(\bar{\omega}_t, \bar{\bar{\omega}} \mid \mathbf{X}) = \frac{2}{r(r-1)I} \sum_{i=1}^{I} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \operatorname{Cov}\left(\frac{1}{I} \sum_{j=1}^{I} \omega_{jt}, \frac{1}{IT} \sum_{k=1}^{I} \sum_{p=-m+1}^{r} \omega_{kp} \mid \mathbf{X} \right)$$

$$= \frac{2}{r(r-1)I^3 T} \sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{k=1}^{I} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \sum_{p=-m+1}^{r} \operatorname{Cov}(\omega_{jt}, \omega_{kp} \mid \mathbf{X})$$

$$= \frac{2}{r(r-1)I^2 T} \sum_{i=1}^{I} \sum_{t=1}^{r} \sum_{p=-m+1}^{r} (r-t) \operatorname{Cov}(\omega_{it}, \omega_{ip} \mid \mathbf{X})$$

$$\frac{2}{r(r-1)I} \sum_{i=1}^{I} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \operatorname{Cov}(\bar{\bar{\omega}}, \omega_{is} \mid \mathbf{X}) = \frac{2}{r(r-1)I} \sum_{i=1}^{I} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \operatorname{Cov}\left(\frac{1}{IT} \sum_{j=1}^{I} \sum_{p=-m+1}^{r} \omega_{jp}, \omega_{is} \mid \mathbf{X} \right)$$

$$= \frac{2}{r(r-1)I^2 T} \sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \sum_{p=-m+1}^{r} \operatorname{Cov}(\omega_{jp}, \omega_{is} \mid \mathbf{X})$$

$$= \frac{2}{r(r-1)I^2 T} \sum_{i=1}^{I} \sum_{t=1}^{r} \sum_{p=-m+1}^{r} (t-1) \operatorname{Cov}(\omega_{it}, \omega_{ip} \mid \mathbf{X})$$

$$\frac{2}{r(r-1)I} \sum_{i=1}^{I} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \operatorname{Cov}(\bar{\bar{\omega}}, \bar{\omega}_i \mid \mathbf{X}) = \frac{2}{r(r-1)I} \sum_{i=1}^{I} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \operatorname{Cov}\left(\frac{1}{IT} \sum_{j=1}^{I} \sum_{p=-m+1}^{r} \omega_{jp}, \frac{1}{T} \sum_{q=-m+1}^{r} \omega_{iq} \mid \mathbf{X} \right)$$

$$= \frac{2}{r(r-1)I^2 T^2} \sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{t=1}^{r-1} \sum_{s=t+1}^{r} \sum_{p=-m+1}^{r} \sum_{q=-m+1}^{r} \operatorname{Cov}(\omega_{jp}, \omega_{iq} \mid \mathbf{X})$$

$$
\begin{aligned}
&= \frac{2}{r(r-1)I^2T^2}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\sum_{p=-m+1}^{r}\mathrm{Var}(\omega_{ip}\mid \mathbf{X}) \\
&\quad + \frac{4}{r(r-1)I^2T^2}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\sum_{p=-m+1}^{r-1}\sum_{q=p+1}^{r}\mathrm{Cov}(\omega_{ip},\omega_{iq}\mid \mathbf{X}) \\
&= \frac{1}{I^2T^2}\sum_{i=1}^{I}\sum_{p=-m+1}^{r}\mathrm{Var}(\omega_{ip}\mid \mathbf{X}) \\
&\quad + \frac{2}{I^2T^2}\left[\sum_{i=1}^{I}\sum_{p=-m+1}^{-1}\sum_{q=p+1}^{0}\mathrm{Cov}(\omega_{ip},\omega_{iq}\mid \mathbf{X})\right. \\
&\qquad + \sum_{i=1}^{I}\sum_{p=-m+1}^{0}\sum_{q=1}^{r}\mathrm{Cov}(\omega_{ip},\omega_{iq}\mid \mathbf{X}) \\
&\qquad \left. + \sum_{i=1}^{I}\sum_{p=1}^{r-1}\sum_{q=p+1}^{r}\mathrm{Cov}(\omega_{ip},\omega_{iq}\mid \mathbf{X})\right] \\
&= \frac{1}{IT}\sigma_\omega^2 + \frac{m(m-1)}{IT^2}\psi^B + \frac{r(r-1)}{IT^2}\psi^A + \frac{2mr}{IT^2}\psi^X
\end{aligned}
$$

$$
\begin{aligned}
\frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}(\bar{\bar{\omega}},\bar{\omega}_s\mid \mathbf{X}) &= \frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}\left(\frac{1}{IT}\sum_{j=1}^{I}\sum_{p=-m+1}^{r}\omega_{jp}, \frac{1}{I}\sum_{k=1}^{I}\omega_{ks}\mid \mathbf{X}\right) \\
&= \frac{2}{r(r-1)I^3T}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{k=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\sum_{p=-m+1}^{r}\mathrm{Cov}(\omega_{jp},\omega_{ks}\mid \mathbf{X}) \\
&= \frac{2}{r(r-1)I^2T}\sum_{i=1}^{I}\sum_{t=1}^{r}\sum_{p=-m+1}^{r}(t-1)\,\mathrm{Cov}(\omega_{it},\omega_{ip}\mid \mathbf{X})
\end{aligned}
$$

$$
\begin{aligned}
\frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}(\bar{\bar{\omega}},\bar{\bar{\omega}}\mid \mathbf{X}) &= \frac{2}{r(r-1)I}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\mathrm{Cov}\left(\frac{1}{IT}\sum_{j=1}^{I}\sum_{p=-m+1}^{r}\omega_{jp},\right. \\
&\qquad \left.\frac{1}{IT}\sum_{k=1}^{I}\sum_{q=_m+1}^{r}\omega_{kq}\mid \mathbf{X}\right) \\
&= \frac{2}{r(r-1)I^3T^2}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{k=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\sum_{p=-m+1}^{r}\sum_{q=-m+1}^{r}\mathrm{Cov}(\omega_{jp},\omega_{kq}\mid \mathbf{X}) \\
&= \frac{2}{r(r-1)I^3T^2}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\sum_{p=-m+1}^{r}\mathrm{Var}(\omega_{jp}\mid \mathbf{X}) \\
&\quad + \frac{4}{r(r-1)I^3T^2}\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{s=t+1}^{r}\sum_{j=1}^{I}\sum_{p=-m+1}^{r-1}\sum_{q=p+1}^{r}\mathrm{Cov}(\omega_{jp},\omega_{jq}\mid \mathbf{X}) \\
&= \frac{1}{I^2T^2}\sum_{j=1}^{I}\sum_{p=-m+1}^{r}\mathrm{Var}(\omega_{jp}\mid \mathbf{X})
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{2}{I^2T^2}\Bigg[\sum_{j=1}^{I}\sum_{p=-m+1}^{r}\sum_{q=-m+1}^{r}\operatorname{Cov}(\omega_{jp},\omega_{jq}\mid \mathbf{X}) \\
&\qquad + \sum_{j=1}^{I}\sum_{p=-m+1}^{r}\sum_{q=-m+1}^{r}\operatorname{Cov}(\omega_{jp},\omega_{jq}\mid \mathbf{X}) \\
&\qquad + \sum_{j=1}^{I}\sum_{p=-m+1}^{r}\sum_{q=-m+1}^{r}\operatorname{Cov}(\omega_{jp},\omega_{jq}\mid \mathbf{X})\Bigg] \\
&= \frac{1}{IT}\sigma_\omega^2 + \frac{m(m-1)}{IT^2}\psi^B + \frac{r(r-1)}{IT^2}\psi^A + \frac{2mr}{IT^2}\psi^X
\end{aligned}
$$

Combining these terms yields

$$
\begin{aligned}
\psi^A_{\hat{\omega}} = \psi^A &- \frac{2}{r(r-1)IT}\sum_{i=1}^{I}\sum_{t=1}^{r}\sum_{p=-m+1}^{r}(r-t)\operatorname{Cov}(\omega_{it},\omega_{ip}\mid \mathbf{X}) - \frac{1}{I}\psi^A \\
&+ \frac{2}{r(r-1)I^2T}\sum_{i=1}^{I}\sum_{t=1}^{r}\sum_{p=-m+1}^{r}(r-t)\operatorname{Cov}(\omega_{it},\omega_{ip}\mid \mathbf{X}) \\
&- \frac{2}{r(r-1)IT}\sum_{i=1}^{I}\sum_{t=1}^{r}\sum_{p=-m+1}^{r}(t-1)\operatorname{Cov}\left(\omega_{it},\omega_{ip}\mid \mathbf{X}\right) \\
&+ \left[\frac{1}{T}\sigma_\omega^2 + \frac{m(m-1)}{T^2}\psi^B + \frac{r(r-1)}{T^2}\psi^A + \frac{2mr}{T^2}\psi^X\right] \\
&+ \frac{2}{r(r-1)I^2T}\sum_{i=1}^{I}\sum_{t=1}^{r}\sum_{p=-m+1}^{r}(t-1)\operatorname{Cov}(\omega_{it},\omega_{ip}\mid \mathbf{X}) \\
&- \left[\frac{1}{IT}\sigma_\omega^2 + \frac{m(m-1)}{IT^2}\psi^B + \frac{r(r-1)}{IT^2}\psi^A + \frac{2mr}{IT^2}\psi^X\right] - \frac{1}{I}\psi^A \\
&+ \frac{2}{r(r-1)I^2T}\sum_{i=1}^{I}\sum_{t=1}^{r}\sum_{p=-m+1}^{r}(r-t)\operatorname{Cov}(\omega_{it},\omega_{ip}\mid \mathbf{X}) + \frac{1}{I}\psi^A \\
&- \frac{2}{r(r-1)I^2T}\sum_{i=1}^{I}\sum_{t=1}^{r}\sum_{p=-m+1}^{r}(r-t)\operatorname{Cov}(\omega_{it},\omega_{ip}\mid \mathbf{X}) \\
&+ \frac{2}{r(r-1)I^2T}\sum_{i=1}^{I}\sum_{t=1}^{r}\sum_{p=-m+1}^{r}(t-1)\operatorname{Cov}(\omega_{it},\omega_{ip}\mid \mathbf{X}) \\
&- \left[\frac{1}{IT}\sigma_\omega^2 + \frac{m(m-1)}{IT^2}\psi^B + \frac{r(r-1)}{IT^2}\psi^A + \frac{2mr}{IT^2}\psi^X\right] \\
&- \frac{2}{r(r-1)I^2T}\sum_{i=1}^{I}\sum_{t=1}^{r}\sum_{p=-m+1}^{r}(t-1)\operatorname{Cov}(\omega_{it},\omega_{ip}\mid \mathbf{X}) \\
&+ \left[\frac{1}{IT}\sigma_\omega^2 + \frac{m(m-1)}{IT^2}\psi^B + \frac{r(r-1)}{IT^2}\psi^A + \frac{2mr}{IT^2}\psi^X\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \psi^A - \frac{2}{r(r-1)IT}\sum_{i=1}^{I}\sum_{t=1}^{r}\sum_{p=-m+1}^{r}(r-1)\operatorname{Cov}(\omega_{it},\omega_{ip}\mid \mathbf{X}) - \frac{1}{I}\psi^A \\
&\quad + \frac{2}{r(r-1)I^2T}\sum_{i=1}^{I}\sum_{t=1}^{r}\sum_{p=-m+1}^{r}(r-1)\operatorname{Cov}(\omega_{it},\omega_{ip}\mid \mathbf{X}) \\
&\quad + \left[\frac{1}{T}\sigma_\omega^2 + \frac{m(m-1)}{T^2}\psi^B + \frac{r(r-1)}{T^2}\psi^A + \frac{2mr}{T^2}\psi^X\right] \\
&\quad - \left[\frac{1}{IT}\sigma_\omega^2 + \frac{m(m-1)}{IT^2}\psi^B + \frac{r(r-1)}{IT^2}\psi^A + \frac{2mr}{IT^2}\psi^X\right] \\
&= \left(\frac{I-1}{I}\right)\psi^A + \left(\frac{I-1}{I}\right)\left[\frac{1}{T}\sigma_\omega^2 + \frac{m(m-1)}{T^2}\psi^B + \frac{r(r-1)}{T^2}\psi^A + \frac{2mr}{T^2}\psi^X\right] \\
&\quad - \frac{2(I-1)}{rI^2T}\sum_{i=1}^{I}\sum_{t=1}^{r}\sum_{p=-m+1}^{r}\operatorname{Cov}(\omega_{it},\omega_{ip}\mid \mathbf{X}) \\
&= \left(\frac{I-1}{I}\right)\psi^A + \left(\frac{I-1}{I}\right)\left[\frac{1}{T}\sigma_\omega^2 + \frac{m(m-1)}{T^2}\psi^B + \frac{r(r-1)}{T^2}\psi^A + \frac{2mr}{T^2}\psi^X\right] \\
&\quad - \frac{2(I-1)}{rI^2T}\left[\sum_{i=1}^{I}\sum_{t=1}^{r}\operatorname{Var}(\omega_{it}\mid \mathbf{X}) + 2\sum_{i=1}^{I}\sum_{t=1}^{r-1}\sum_{p=t+1}^{r}\operatorname{Cov}(\omega_{it},\omega_{ip}\mid \mathbf{X}) + \sum_{i=1}^{I}\sum_{t=1}^{r}\sum_{p=-m+1}^{0}\operatorname{Cov}(\omega_{it},\omega_{ip}\mid \mathbf{X})\right] \\
&= \left(\frac{I-1}{I}\right)\psi^A + \left(\frac{I-1}{I}\right)\left[\frac{1}{T}\sigma_\omega^2 + \frac{m(m-1)}{T^2}\psi^B + \frac{r(r-1)}{T^2}\psi^A + \frac{2mr}{T^2}\psi^X\right] \\
&\quad - \left(\frac{I-1}{I}\mid\right)\left[\frac{2}{T}\sigma_\omega^2 + \frac{2(r-1)}{T}\psi^A + \frac{2m}{T}\psi^X\right] \\
&= \left(\frac{I-1}{I}\right)\left[\frac{1}{T} - \frac{2}{T}\right]\sigma_\omega^2 + \left(\frac{I-1}{I}\right)\left[\frac{m(m-1)}{T^2}\right]\psi^B \\
&\quad + \left(\frac{I-1}{I}\right)\left[1 + \frac{r(r-1)}{T^2} - \frac{2(r-2)}{T}\right]\psi^A + \left(\frac{I-1}{I}\right)\left[\frac{2mr}{T^2} + \frac{2m}{T}\right]\psi^X \\
&= -\left(\frac{I-1}{IT}\right)\sigma_\omega^2 + \left(\frac{(I-1)m(m-1)}{IT^2}\right)\psi^B \\
&\quad + \left(\frac{(I-1)(m^2+2m+r)}{IT^2}\right)\psi^A - \left(\frac{2(I-1)m^2}{IT^2}\right)\psi^X
\end{aligned}
$$

By symmetry, the expression for $\psi^B_{\hat{\omega}}$ is

$$
\begin{aligned}
\psi^B_{\hat{\omega}} &= -\left(\frac{I-1}{IT}\right)\sigma_\omega^2 + \left(\frac{(I-1)(r^2+2r+m)}{IT^2}\right)\psi^B \\
&\quad + \left(\frac{(I-1)r(r-1)}{IT^2}\right)\psi^A - \left(\frac{2(I-1)r^2}{IT^2}\right)\psi^X
\end{aligned}
$$

We finally derive an expression for $\psi^X_{\hat{\omega}}$.

$$\begin{aligned}
\psi^X_{\hat{\omega}} &= \frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\text{Cov}(\hat{\omega}_{it},\hat{\omega}_{is}\mid \mathbf{X}) \\
&= \frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\text{Cov}(\omega_{it}-\bar{\omega}_i-\bar{\omega}_t+\bar{\bar{\omega}},\omega_{is}-\bar{\omega}_i-\bar{\omega}_s+\bar{\bar{\omega}}\mid \mathbf{X}) \\
&= \frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}[\text{Cov}(\omega_{it},\omega_{is}\mid \mathbf{X})-\text{Cov}(\omega_{it},\bar{\omega}_i\mid \mathbf{X})-\text{Cov}(\omega_{it},\bar{\omega}_s\mid \mathbf{X})+\text{Cov}(\omega_{it},\bar{\bar{\omega}}\mid \mathbf{X})] \\
&\qquad -\text{Cov}(\bar{\omega}_i,\omega_{is}\mid \mathbf{X})+\text{Cov}(\bar{\omega}_i,\bar{\omega}_i\mid \mathbf{X})+\text{Cov}(\bar{\omega}_i,\bar{\omega}_s\mid \mathbf{X})-\text{Cov}(\bar{\omega}_i,\bar{\bar{\omega}}\mid \mathbf{X}) \\
&\qquad -\text{Cov}(\bar{\omega}_t,\omega_{is}\mid \mathbf{X})+\text{Cov}(\bar{\omega}_t,\bar{\omega}_i\mid \mathbf{X})+\text{Cov}(\bar{\omega}_t,\bar{\omega}_s\mid \mathbf{X})-\text{Cov}(\bar{\omega}_t,\bar{\bar{\omega}}\mid \mathbf{X}) \\
&\qquad +\text{Cov}(\bar{\bar{\omega}},\omega_{is}\mid \mathbf{X})-\text{Cov}(\bar{\bar{\omega}},\bar{\omega}_i\mid \mathbf{X})-\text{Cov}(\bar{\bar{\omega}},\bar{\omega}_s\mid \mathbf{X})+\text{Cov}(\bar{\bar{\omega}},\bar{\bar{\omega}}\mid \mathbf{X})]
\end{aligned}$$

We again calculate each term.

$$\frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\text{Cov}(\omega_{is},\omega_{it}\mid \mathbf{X}) = \psi^X$$

$$\begin{aligned}
\frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\text{Cov}(\omega_{it},\bar{\omega}_i\mid \mathbf{X}) &= \frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\text{Cov}\left(\omega_{it},\frac{1}{T}\sum_{p=-m+1}^{r}\omega_{ip}\mid \mathbf{X}\right) \\
&= \frac{1}{ITmr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\sum_{p=-m+1}^{r}\text{Cov}(\omega_{it},\omega_{ip}\mid \mathbf{X}) \\
&= \frac{1}{ITm}\Bigg[\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\text{Var}(\omega_{it}\mid \mathbf{X}) \\
&\qquad +2\sum_{i=1}^{T}\sum_{t=-m+1}^{-1}\sum_{p=t+1}^{0}\text{Cov}(\omega_{it},\omega_{ip}\mid \mathbf{X}) \\
&\qquad +\sum_{i=1}^{T}\sum_{t=-m+1}^{0}\sum_{p=1}^{r}\text{Cov}(\omega_{it},\omega_{ip}\mid \mathbf{X})\Bigg] \\
&= \frac{1}{T}\sigma^2_\omega+\frac{m-1}{T}\psi^B+\frac{r}{T}\psi^X
\end{aligned}$$

$$\begin{aligned}
\frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\text{Cov}(\omega_{it},\bar{\omega}_s\mid \mathbf{X}) &= \frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\text{Cov}\left(\omega_{it},\frac{1}{I}\sum_{j=1}^{I}\omega_{js}\mid \mathbf{X}\right) \\
&= \frac{1}{I^2mr}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\text{Cov}(\omega_{it},\omega_{js}\mid \mathbf{X}) \\
&= \frac{1}{I^2mr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\text{Cov}(\omega_{it},\omega_{is}\mid \mathbf{X}) \\
&= \frac{1}{I}\psi^X
\end{aligned}$$

$$
\begin{aligned}
\frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}(\omega_{it},\bar{\bar{\omega}}\mid\mathbf{X}) &= \frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}\left(\omega_{it},\frac{1}{IT}\sum_{j=1}^{I}\sum_{p=-m+1}^{r}\omega_{jp}\mid\mathbf{X}\right)\\
&= \frac{1}{I^2Tmr}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\sum_{p=-m+1}^{r}\mathrm{Cov}(\omega_{it},\omega_{jp}\mid\mathbf{X})\\
&= \frac{1}{I^2Tm}\sum_{j=1}^{I}\sum_{t=-m+1}^{0}\sum_{p=-m+1}^{r}\mathrm{Cov}(\omega_{it},\omega_{ip}\mid\mathbf{X})\\
&= \frac{1}{I^2Tm}\Bigg[\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\mathrm{Var}(\omega_{it}\mid\mathbf{X})\\
&\qquad+2\sum_{i=1}^{I}\sum_{t=-m+1}^{-1}\sum_{p=t+1}^{0}\mathrm{Cov}(\omega_{it},\omega_{ip}\mid\mathbf{X})\\
&\qquad+\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{p=1}^{r}\mathrm{Cov}(\omega_{it},\omega_{ip}\mid\mathbf{X})\Bigg]\\
&= \frac{1}{IT}\sigma_\omega^2+\frac{m-1}{IT}\psi^B+\frac{r}{IT}\psi^X
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}(\bar{\omega}_i,\omega_{is}\mid\mathbf{X}) &= \frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}\left(\frac{1}{T}\sum_{p=-m+1}^{r}\omega_{ip},\omega_{is}\mid\mathbf{X}\right)\\
&= \frac{1}{ITmr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\sum_{p=-m+1}^{r}\mathrm{Cov}(\omega_{ip},\omega_{is}\mid\mathbf{X})\\
&= \frac{1}{ITr}\Bigg[\sum_{i=1}^{I}\sum_{s=1}^{r}\mathrm{Var}(\omega_{is}\mid\mathbf{X})\\
&\qquad+\sum_{i=1}^{I}\sum_{s=1}^{r}\sum_{p=-m+1}^{0}\mathrm{Cov}(\omega_{is},\omega_{ip}\mid\mathbf{X})\\
&\qquad+2\sum_{i=1}^{I}\sum_{s=1}^{r-1}\sum_{p=s+1}^{r}\mathrm{Cov}(\omega_{is},\omega_{ip}\mid\mathbf{X})\Bigg]\\
&= \frac{1}{T}\sigma_\omega^2+\frac{r-1}{T}\psi^A+\frac{m}{T}\psi^X
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}(\bar{\omega}_i,\bar{\omega}_i\mid\mathbf{X}) &= \frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}\left(\frac{1}{T}\sum_{p=-m+1}^{r}\omega_{ip},\frac{1}{T}\sum_{q=-m+1}^{r}\omega_{iq}\mid\mathbf{X}\right)\\
&= \frac{1}{IT^2mr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\sum_{p=-m+1}^{r}\sum_{q=-m+1}^{r}\mathrm{Cov}(\omega_{ip},\omega_{iq}\mid\mathbf{X})\\
&= \frac{1}{IT^2mr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\sum_{p=-m+1}^{r}\mathrm{Var}(\omega_{ip}\mid\mathbf{X})\\
&\quad+\frac{2}{IT^2mr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\sum_{p=-m+1}^{r-1}\sum_{q=p+1}^{r}\mathrm{Cov}(\omega_{ip},\omega_{iq}\mid\mathbf{X})
\end{aligned}
$$

$$= \frac{1}{IT^2} \sum_{i=1}^{I} \sum_{p=-m+1}^{r} \mathrm{Var}(\omega_{ip} \mid \mathbf{X})$$

$$+ \frac{2}{IT^2} \Bigg[\sum_{i=1}^{I} \sum_{p=-m+1}^{-1} \sum_{q=p+1}^{0} \mathrm{Cov}(\omega_{ip}, \omega_{iq} \mid \mathbf{X})$$

$$+ \sum_{i=1}^{I} \sum_{p=-m+1}^{0} \sum_{q=1}^{r} \mathrm{Cov}(\omega_{ip}, \omega_{iq} \mid \mathbf{X})$$

$$+ \sum_{i=1}^{I} \sum_{p=1}^{r-1} \sum_{q=p+1}^{r} \mathrm{Cov}(\omega_{ip}, \omega_{iq} \mid \mathbf{X}) \Bigg]$$

$$= \frac{1}{T}\sigma_\omega^2 + \frac{m(m-1)}{T^2}\psi^B + \frac{r(r-1)}{T^2}\psi^A + \frac{2mr}{T^2}\psi^X$$

$$\frac{1}{Imr} \sum_{i=1}^{I} \sum_{t=-m+1}^{0} \sum_{s=1}^{r} \mathrm{Cov}(\bar{\omega}_i, \bar{\omega}_s \mid \mathbf{X}) = \frac{1}{Imr} \sum_{i=1}^{I} \sum_{t=-m+1}^{0} \sum_{s=1}^{r} \mathrm{Cov}\left(\frac{1}{T} \sum_{p=-m+1}^{r} \omega_{ip}, \frac{1}{I} \sum_{j=1}^{I} \omega_{js} \mid \mathbf{X} \right)$$

$$= \frac{1}{I^2Tmr} \sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{t=-m+1}^{0} \sum_{s=1}^{r} \sum_{p=-m+1}^{r} \mathrm{Cov}(\omega_{ip}, \omega_{js} \mid \mathbf{X})$$

$$= \frac{1}{I^2Tr} \Bigg[\sum_{i=1}^{I} \sum_{s=1}^{r} \mathrm{Var}(\omega_{is} \mid \mathbf{X})$$

$$+ \sum_{i=1}^{I} \sum_{s=1}^{r} \sum_{p=-m+1}^{0} \mathrm{Cov}(\omega_{is}, \omega_{ip} \mid \mathbf{X})$$

$$+2 \sum_{i=1}^{I} \sum_{s=1}^{r-1} \sum_{p=s+1}^{r} \mathrm{Cov}(\omega_{is}, \omega_{ip} \mid \mathbf{X}) \Bigg]$$

$$= \frac{1}{IT}\sigma_\omega^2 + \frac{r-1}{IT}\psi^A + \frac{m}{IT}\psi^X$$

$$\frac{1}{Imr} \sum_{i=1}^{I} \sum_{t=-m+1}^{0} \sum_{s=1}^{r} \mathrm{Cov}(\bar{\omega}_i, \bar{\bar{\omega}} \mid \mathbf{X}) = \frac{1}{Imr} \sum_{i=1}^{I} \sum_{t=-m+1}^{0} \sum_{s=1}^{r} \mathrm{Cov}\left(\frac{1}{T} \sum_{p=-m+1}^{r} \omega_{ip}, \frac{1}{IT} \sum_{j=1}^{I} \sum_{q=-m+1}^{r} \omega_{jq} \mid \mathbf{X} \right)$$

$$= \frac{1}{I^2T^2mr} \sum_{i=1}^{I} \sum_{j=1}^{I} \sum_{t=-m+1}^{0} \sum_{s=1}^{r} \sum_{p=-m+1}^{r} \sum_{q=-m+1}^{r} \mathrm{Cov}(\omega_{ip}, \omega_{jq} \mid \mathbf{X})$$

$$= \frac{1}{I^2T^2mr} \sum_{i=1}^{I} \sum_{t=-m+1}^{0} \sum_{s=1}^{r} \sum_{p=-m+1}^{r} \mathrm{Var}(\omega_{ip} \mid \mathbf{X})$$

$$+ \frac{2}{I^2T^2mr} \sum_{i=1}^{I} \sum_{t=-m+1}^{0} \sum_{s=1}^{r} \sum_{p=-m+1}^{r-1} \sum_{q=p+1}^{r} \mathrm{Cov}(\omega_{ip}, \omega_{iq} \mid \mathbf{X})$$

$$= \frac{1}{I^2T^2} \sum_{i=1}^{I} \sum_{p=-m+1}^{r} \mathrm{Var}(\omega_{ip} \mid \mathbf{X})$$

$$+ \frac{2}{I^2T^2} \Bigg[\sum_{i=1}^{I} \sum_{p=-m+1}^{-1} \sum_{q=p+1}^{0} \mathrm{Cov}(\omega_{ip}, \omega_{iq} \mid \mathbf{X})$$

$$
\begin{aligned}
&+\sum_{i=1}^{I}\sum_{p=-m+1}^{0}\sum_{q=1}^{r}\mathrm{Cov}(\omega_{ip},\omega_{iq}\mid\mathbf{X}) \\
&+\sum_{i=1}^{I}\sum_{p=1}^{r-1}\sum_{q=p+1}^{r}\mathrm{Cov}(\omega_{ip},\omega_{iq}\mid\mathbf{X})\Bigg] \\
&=\frac{1}{IT}\sigma_\omega^2+\frac{m(m-1)}{IT^2}\psi^B+\frac{r(r-1)}{IT^2}\psi^A+\frac{2mr}{IT^2}\psi^X
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}(\bar\omega_t,\omega_{is}\mid\mathbf{X}) &= \frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}\left(\frac{1}{I}\sum_{j=1}^{I}\omega_{jt},\omega_{is}\mid\mathbf{X}\right) \\
&= \frac{1}{I^2mr}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}(\omega_{jt},\omega_{is}\mid\mathbf{X}) \\
&= \frac{1}{I^2mr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}(\omega_{it},\omega_{is}\mid\mathbf{X}) \\
&= \frac{1}{I}\psi^X
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}(\bar\omega_t,\bar\omega_i\mid\mathbf{X}) &= \frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}\left(\frac{1}{I}\sum_{j=1}^{I}\omega_{jt},\frac{1}{T}\sum_{p=-m+1}^{r}\omega_{ip}\mid\mathbf{X}\right) \\
&= \frac{1}{I^2Tmr}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\sum_{p=-m+1}^{r}\mathrm{Cov}(\omega_{jt},\omega_{ip}\mid\mathbf{X}) \\
&= \frac{1}{I^2Tm}\Bigg[\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\mathrm{Var}(\omega_{it}\mid\mathbf{X}) \\
&\quad +2\sum_{i=1}^{I}\sum_{t=-m+1}^{-1}\sum_{p=t+1}^{0}\mathrm{Cov}(\omega_{it},\omega_{ip}\mid\mathbf{X}) \\
&\quad +\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{p=1}^{r}\mathrm{Cov}(\omega_{it},\omega_{ip}\mid\mathbf{X})\Bigg] \\
&= \frac{1}{IT}\sigma_\omega^2+\frac{m-1}{IT}\psi^B+\frac{r}{IT}\psi^X
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}(\bar\omega_t,\bar\omega_s\mid\mathbf{X}) &= \frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}\left(\frac{1}{I}\sum_{j=1}^{I}\omega_{jt},\frac{1}{I}\sum_{k=1}^{I}\omega_{ks}\mid\mathbf{X}\right) \\
&= \frac{1}{I^3mr}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{k=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}(\omega_{jt},\omega_{ks}\mid\mathbf{X}) \\
&= \frac{1}{I^2mr}\sum_{j=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}(\omega_{jt},\omega_{js}\mid\mathbf{X}) \\
&= \frac{1}{I}\psi^X
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}(\bar{\omega}_t,\bar{\bar{\omega}}\mid \mathbf{X}) &= \frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}\left(\frac{1}{I}\sum_{j=1}^{I}\omega_{jt},\frac{1}{IT}\sum_{k=1}^{I}\sum_{p=-m+1}\omega_{kp}\mid \mathbf{X}\right)\\
&= \frac{1}{I^3Tmr}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{k=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\sum_{p=-m+1}^{r}\mathrm{Cov}(\omega_{jt},\omega_{kp}\mid \mathbf{X})\\
&= \frac{1}{I^2Tm}\left[\sum_{j=1}^{I}\sum_{t=-m+1}^{0}\mathrm{Var}(\omega_{jt}\mid \mathbf{X})\right.\\
&\qquad + 2\sum_{j=1}^{I}\sum_{t=-m+1}^{-1}\sum_{p=t+1}^{0}\mathrm{Cov}(\omega_{jt},\omega_{jp}\mid \mathbf{X})\\
&\qquad \left.+\sum_{j=1}^{I}\sum_{t=-m+1}^{0}\sum_{p=1}^{r}\mathrm{Cov}(\omega_{jt},\omega_{jp}\mid \mathbf{X})\right]\\
&= \frac{1}{IT}\sigma_\omega^2+\frac{m-1}{IT}\psi^B+\frac{r}{IT}\psi^X\\
\frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}(\bar{\bar{\omega}},\omega_{is}\mid \mathbf{X}) &= \frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}\left(\frac{1}{IT}\sum_{j=1}^{I}\sum_{p=-m+1}^{r}\omega_{jp},\omega_{is}\mid \mathbf{X}\right)\\
&= \frac{1}{I^2Tmr}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\sum_{p=-m+1}^{r}\mathrm{Cov}(\omega_{jp},\omega_{is}\mid \mathbf{X})\\
&= \frac{1}{I^2Tr}\left[\sum_{i=1}^{I}\sum_{s=1}^{r}\mathrm{Var}(\omega_{is}\mid \mathbf{X})\right.\\
&\qquad +\sum_{i=1}^{I}\sum_{s=1}^{r}\sum_{p=-m+1}^{0}\mathrm{Cov}(\omega_{is},\omega_{ip}\mid \mathbf{X})\\
&\qquad \left.+2\sum_{i=1}^{I}\sum_{s=1}^{r-1}\sum_{p=s+1}^{r}\mathrm{Cov}(\omega_{is},\omega_{ip}\mid \mathbf{X})\right]\\
&= \frac{1}{IT}\sigma_\omega^2+\frac{r-1}{IT}\psi^A+\frac{m}{IT}\psi^X\\
\frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}(\bar{\bar{\omega}},\bar{\omega}_i\mid \mathbf{X}) &= \frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\mathrm{Cov}\left(\frac{1}{IT}\sum_{j=1}^{I}\sum_{p=-m+1}^{r}\omega_{jp},\frac{1}{T}\sum_{q=-m+1}^{r}\omega_{iq}\mid \mathbf{X}\right)\\
&= \frac{1}{I^2T^2mr}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\sum_{p=-m+1}^{r}\sum_{q=-m+1}^{r}\mathrm{Cov}(\omega_{jp},\omega_{iq}\mid \mathbf{X})\\
&= \frac{1}{I^2T^2mr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\sum_{p=-m+1}^{r}\mathrm{Var}(\omega_{ip}\mid \mathbf{X})\\
&\quad +\frac{2}{I^2T^2mr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\sum_{p=-m+1}^{r-1}\sum_{q=p+1}^{r}\mathrm{Cov}(\omega_{jp},\omega_{iq}\mid \mathbf{X})\\
&= \frac{1}{I^2T^2}\sum_{i=1}^{I}\sum_{p=-m+1}^{r}\mathrm{Var}(\omega_{ip}\mid \mathbf{X})
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{2}{I^2T^2}\left[\sum_{i=1}^{I}\sum_{p=-m+1}^{-1}\sum_{q=p+1}^{0}\operatorname{Cov}(\omega_{jp},\omega_{iq}\mid \mathbf{X})\right.\\
&\qquad + \sum_{i=1}^{I}\sum_{p=-m+1}^{0}\sum_{q=1}^{r}\operatorname{Cov}(\omega_{jp},\omega_{iq}\mid \mathbf{X})\\
&\qquad \left.+ \sum_{i=1}^{I}\sum_{p=1}^{r-1}\sum_{q=p+1}^{r}\operatorname{Cov}(\omega_{jp},\omega_{iq}\mid \mathbf{X})\right]\\
&= \frac{1}{IT}\sigma_\omega^2 + \frac{m(m-1)}{IT^2}\psi^B + \frac{r(r-1)}{IT^2}\psi^A + \frac{2mr}{IT^2}\psi^X
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\operatorname{Cov}(\bar{\bar{\omega}},\bar{\omega}_s\mid \mathbf{X}) &= \frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\operatorname{Cov}\left(\frac{1}{IT}\sum_{j=1}^{I}\sum_{p=-m+1}^{r}\omega_{jp}, \frac{1}{I}\sum_{k=1}^{I}\omega_{ks}\mid \mathbf{X}\right)\\
&= \frac{1}{I^3Tmr}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{k=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\sum_{p=-m+1}^{r}\operatorname{Cov}(\omega_{jp},\omega_{ks}\mid \mathbf{X})\\
&= \frac{1}{I^2Tr}\left[\sum_{j=1}^{I}\sum_{s=1}^{r}\operatorname{Var}(\omega_{js}\mid \mathbf{X})\right.\\
&\qquad + \sum_{j=1}^{I}\sum_{s=1}^{r}\sum_{p=-m+1}^{0}\operatorname{Cov}(\omega_{js},\omega_{jp}\mid \mathbf{X})\\
&\qquad \left.+2\sum_{j=1}^{I}\sum_{s=1}^{r-1}\sum_{p=s+1}^{r}\operatorname{Cov}(\omega_{js},\omega_{jp}\mid \mathbf{X})\right]\\
&= \frac{1}{IT}\sigma_\omega^2 + \frac{r-1}{IT}\psi^A + \frac{m}{IT}\psi^X
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\operatorname{Cov}(\bar{\bar{\omega}},\bar{\bar{\omega}}\mid \mathbf{X}) &= \frac{1}{Imr}\sum_{i=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\operatorname{Cov}\left(\frac{1}{IT}\sum_{j=1}^{I}\sum_{p=-m+1}^{r}\omega_{jp}, \frac{1}{IT}\sum_{k=1}^{I}\sum_{q=-m+1}^{r}\omega_{kq}\mid \mathbf{X}\right)\\
&= \frac{1}{I^3T^2mr}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{k=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\sum_{p=-m+1}^{r}\sum_{q=-m+1}^{r}\operatorname{Cov}(\omega_{jp},\omega_{kq}\mid \mathbf{X})\\
&= \frac{1}{I^3T^2mr}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\sum_{p=-m+1}^{r}\operatorname{Var}(\omega_{jp}\mid \mathbf{X})\\
&\quad + \frac{2}{I^3T^2mr}\sum_{i=1}^{I}\sum_{j=1}^{I}\sum_{t=-m+1}^{0}\sum_{s=1}^{r}\sum_{p=-m+1}^{r-1}\sum_{q=p+1}^{r}\operatorname{Cov}(\omega_{jp},\omega_{jq}\mid \mathbf{X})\\
&= \frac{1}{I^2T^2}\sum_{j=1}^{I}\sum_{p=-m+1}^{r}\operatorname{Var}(\omega_{jp}\mid \mathbf{X})\\
&\quad + \frac{2}{I^2T^2}\left[\sum_{j=1}^{I}\sum_{p=-m+1}^{-1}\sum_{q=p+1}^{0}\operatorname{Cov}(\omega_{jp},\omega_{jq}\mid \mathbf{X})\right.\\
&\qquad + \sum_{j=1}^{I}\sum_{p=-m+1}^{0}\sum_{q=1}^{r}\operatorname{Cov}(\omega_{jp},\omega_{jq}\mid \mathbf{X})
\end{aligned}
$$

$$
\left. + \sum_{j=1}^{I} \sum_{p=1}^{r-1} \sum_{q=p+1}^{r} \mathrm{Cov}(\omega_{jp}, \omega_{jq} \mid \mathbf{X}) \right]
$$

$$
= \frac{1}{IT}\sigma_\omega^2 + \frac{m(m-1)}{IT^2}\psi^B + \frac{r(r-1)}{IT^2}\psi^A + \frac{2mr}{IT^2}\psi^X
$$

Combining these terms yields

$$
\begin{aligned}
\psi_{\hat{\omega}}^X = \psi^X &- \left[\frac{1}{T}\sigma_\omega^2 + \frac{m-1}{T}\psi^B + \frac{r}{T}\psi^X\right] - \frac{1}{I}\psi^X + \left[\frac{1}{IT}\sigma_\omega^2 + \frac{m-1}{IT}\psi^B + \frac{r}{IT}\psi^X\right] \\
&- \left[\frac{1}{T}\sigma_\omega^2 + \frac{r-1}{T}\psi^A + \frac{m}{T}\psi^X\right] + \left[\frac{1}{T}\sigma_\omega^2 + \frac{m(m-1)}{T^2}\psi^B + \frac{r(r-1)}{T^2}\psi^A + \frac{2mr}{T^2}\psi^X\right] \\
&+ \left[\frac{1}{IT}\sigma_\omega^2 + \frac{r-1}{IT}\psi^A + \frac{m}{IT}\psi^X\right] - \left[\frac{1}{IT}\sigma_\omega^2 + \frac{m(m-1)}{IT^2}\psi^B + \frac{r(r-1)}{IT^2}\psi^A + \frac{2mr}{IT^2}\psi^X\right] \\
&- \frac{1}{I}\psi^X + \left[\frac{1}{IT}\sigma_\omega^2 + \frac{m-1}{IT}\psi^B + \frac{r}{IT}\psi^X\right] + \frac{1}{I}\psi^X - \left[\frac{1}{IT}\sigma_\omega^2 + \frac{m-1}{IT}\psi^B + \frac{r}{IT}\psi^X\right] \\
&+ \left[\frac{1}{IT}\sigma_\omega^2 + \frac{r-1}{IT}\psi^A + \frac{m}{IT}\psi^X\right] - \left[\frac{1}{IT}\sigma_\omega^2 + \frac{m(m-1)}{IT^2}\psi^B + \frac{r(r-1)}{IT^2}\psi^A + \frac{2mr}{IT^2}\psi^X\right] \\
&- \left[\frac{1}{IT}\sigma_\omega^2 + \frac{r-1}{IT}\psi^A + \frac{m}{IT}\psi^X\right] + \left[\frac{1}{IT}\sigma_\omega^2 + \frac{m(m-1)}{IT^2}\psi^B + \frac{r(r-1)}{IT^2}\psi^A + \frac{2mr}{IT^2}\psi^X\right] \\
= \psi^X &- \left[\frac{1}{T}\sigma_\omega^2 + \frac{m-1}{T}\psi^B + \frac{r}{T}\psi^X\right] - \frac{1}{I}\psi^X + \left[\frac{1}{IT}\sigma_\omega^2 + \frac{m-1}{IT}\psi^B + \frac{r}{IT}\psi^X\right] \\
&- \left[\frac{1}{T}\sigma_\omega^2 + \frac{r-1}{T}\psi^A + \frac{m}{T}\psi^X\right] + \left[\frac{1}{T}\sigma_\omega^2 + \frac{m(m-1)}{T^2}\psi^B + \frac{r(r-1)}{T^2}\psi^A + \frac{2mr}{T^2}\psi^X\right] \\
&+ \left[\frac{1}{IT}\sigma_\omega^2 + \frac{r-1}{IT}\psi^A + \frac{m}{IT}\psi^X\right] - \left[\frac{1}{IT}\sigma_\omega^2 + \frac{m(m-1)}{IT^2}\psi^B + \frac{r(r-1)}{IT^2}\psi^A + \frac{2mr}{IT^2}\psi^X\right] \\
= &\left[-\frac{1}{T} + \frac{1}{IT} - \frac{1}{T} + \frac{1}{T} + \frac{1}{IT} - \frac{1}{IT}\right]\sigma_\omega^2 \\
&+ \left[-\frac{m-1}{T} + \frac{m-1}{IT} + \frac{m(m-1)}{T^2} - \frac{m(m-1)}{IT^2}\right]\psi^B \\
&+ \left[-\frac{r-1}{T} + \frac{r(r-1)}{T^2} + \frac{r-1}{IT} + \frac{r(r-1)}{IT^2}\right]\psi^A \\
&+ \left[1 - \frac{r}{T} - \frac{1}{I} + \frac{r}{IT} - \frac{m}{T} + \frac{2mr}{T^2} + \frac{m}{IT} - \frac{2mr}{IT^2}\right]\psi^X \\
\\
= &-\left(\frac{I-1}{IT}\right)\sigma_\omega^2 - \left(\frac{(I-1)r(m-1)}{IT^2}\right)\psi^B \\
&- \left(\frac{(I-1)m(r-1)}{IT^2}\right)\psi^A + \left(\frac{2(I-1)mr}{IT^2}\right)\psi^X
\end{aligned}
$$

To summarize, the residual-based variance and covariance parameters can be expressed as functions of the variance and covariance parameters that define the error structure.

$$\sigma^2_{\hat{\omega}} = \left(\frac{(I-1)(T-1)}{IT}\right)\sigma^2_\omega - \left(\frac{(I-1)m(m-1)}{IT^2}\right)\psi^B - \left(\frac{(I-1)r(r-1)}{IT^2}\right)\psi^A - \left(\frac{2(I-1)mr}{IT^2}\right)\psi^X$$
$$\psi^B_{\hat{\omega}} = -\left(\frac{I-1}{IT}\right)\sigma^2_\omega + \left(\frac{(I-1)(r^2+2r+m)}{IT^2}\right)\psi^B + \left(\frac{(I-1)r(r-1)}{IT^2}\right)\psi^A - \left(\frac{2(I-1)r^2}{IT^2}\right)\psi^X$$
$$\psi^A_{\hat{\omega}} = -\left(\frac{I-1}{IT}\right)\sigma^2_\omega + \left(\frac{(I-1)m(m-1)}{IT^2}\right)\psi^B + \left(\frac{(I-1)(m^2+2m+r)}{IT^2}\right)\psi^A - \left(\frac{2(I-1)m^2}{IT^2}\right)\psi^X$$
$$\psi^X_{\hat{\omega}} = -\left(\frac{I-1}{IT}\right)\sigma^2_\omega - \left(\frac{(I-1)r(m-1)}{IT^2}\right)\psi^B - \left(\frac{(I-1)m(r-1)}{IT^2}\right)\psi^A + \left(\frac{2(I-1)mr}{IT^2}\right)\psi^X$$

Factoring $\frac{I-1}{IT^2}$ from each term, we have:

$$\sigma^2_{\hat{\omega}} = \left(\frac{I-1}{IT^2}\right)\Big(T(T-1)\sigma^2_\omega - m(m-1)\psi^B - r(r-1)\psi^A - 2mr\psi^X\Big)$$
$$\psi^B_{\hat{\omega}} = \left(\frac{I-1}{IT^2}\right)\Big(-T\sigma^2_\omega + (r^2+2r+m)\psi^B + r(r-1)\psi^A - 2r^2\psi^X\Big)$$
$$\psi^A_{\hat{\omega}} = \left(\frac{I-1}{IT^2}\right)\Big(-T\sigma^2_\omega + m(m-1)\psi^B + (m^2+2m+r)\psi^A - 2m^2\psi^X\Big)$$
$$\psi^X_{\hat{\omega}} = \left(\frac{I-1}{IT^2}\right)\Big(-T\sigma^2_\omega - r(m-1)\psi^B - m(r-1)\psi^A + 2mr\psi^X\Big)$$

In matrix notation:

$$\begin{bmatrix}\sigma^2_{\hat{\omega}}\\ \psi^B_{\hat{\omega}}\\ \psi^A_{\hat{\omega}}\\ \psi^X_{\hat{\omega}}\end{bmatrix} = \mathbf{\Gamma}\begin{bmatrix}\sigma^2_\omega\\ \psi^B\\ \psi^A\\ \psi^X\end{bmatrix}$$

where

$$\mathbf{\Gamma} = \frac{I-1}{IT^2}\begin{bmatrix} T(T-1) & -m(m-1) & -r(r-1) & -2mr\\ -T & r^2+2r+m & r(r-1) & -2r^2\\ -T & m(m-1) & m^2+2m+r & -2m^2\\ -T & -r(m-1) & -m(r-1) & 2mr\end{bmatrix}$$

Minimum detectable effect We are ultimately interested in deriving an expression for the MDE of an experiment as a function of the residual-based parameters, $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$, rather than the true parameters, σ^2_ω, ψ^B, ψ^A, and ψ^X. Recall that:

$$MDE = \left(t^J_{1-\kappa} + t^J_{\alpha/2} \mid \mathbf{X}\right)\sqrt{\left(\frac{1}{P(1-P)J} \mid \mathbf{X}\right)\left[\left(\frac{m+r}{mr}\right)\sigma^2_\omega + \left(\frac{m-1}{m}\right)\psi^B + \left(\frac{r-1}{r}\right)\psi^A - 2\psi^X\right]}$$

Having solved for $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$ as linear functions of the true parameters σ^2_ω, ψ^B, ψ^A, and ψ^X, we can define k_σ, k_B, k_A, and k_X as coefficients on the residual-based parameters $\sigma^2_{\hat{\omega}}$, $\psi^B_{\hat{\omega}}$, $\psi^A_{\hat{\omega}}$, and $\psi^X_{\hat{\omega}}$. These coefficients will allow us to use residual-based parameters in the SCR formula in place of the true parameters. In other words, k_σ, k_B, k_A, and k_X must satisfy the following equation:[28]

$$\begin{aligned}&\left(t^J_{1-\kappa}+t^J_{\alpha/2}\right)\sqrt{\left(\frac{1}{P(1-P)J}\right)\left[\left(\frac{m+r}{mr}\right)k_\sigma\sigma^2_{\hat\omega}+\left(\frac{m-1}{m}\right)k_B\psi^B_{\hat\omega}+\left(\frac{r-1}{r}\right)k_A\psi^A_{\hat\omega}-2k_X\psi^X_{\hat\omega}\right]}\\&\quad=\left(t^J_{1-\kappa}+t^J_{\alpha/2}\right)\sqrt{\left(\frac{1}{P(1-P)J}\right)\left[\left(\frac{m+r}{mr}\right)\sigma^2_{\omega}+\left(\frac{m-1}{m}\right)\psi^B+\left(\frac{r-1}{r}\right)\psi^A-2\psi^X\right]}\end{aligned}$$

or more simply

$$\begin{aligned}&\left(\frac{m+r}{mr}\right)k_\sigma\sigma^2_{\hat\omega}+\left(\frac{m-1}{m}\right)k_B\psi^B_{\hat\omega}+\left(\frac{r-1}{r}\right)k_A\psi^A_{\hat\omega}-2k_X\psi^X_{\hat\omega}\\&\quad=\left(\frac{m+r}{mr}\right)\sigma^2_{\omega}+\left(\frac{m-1}{m}\right)\psi^B+\left(\frac{r-1}{r}\right)\psi^A-2\psi^X\end{aligned}$$

We can express this equality in matrix form as[29]

$$\begin{aligned}\begin{bmatrix}\frac{m+r}{mr} & \frac{m-1}{m} & \frac{r-1}{r} & -2\end{bmatrix}\begin{bmatrix}\sigma^2_\omega\\ \psi^B\\ \psi^A\\ \psi^X\end{bmatrix}&=\begin{bmatrix}\left(\frac{m+r}{mr}\right)k_\sigma & \left(\frac{m-1}{m}\right)k_B & \left(\frac{r-1}{r}\right)k_A & -2k_X\end{bmatrix}\begin{bmatrix}\sigma^2_{\hat\omega}\\ \psi^B_{\hat\omega}\\ \psi^A_{\hat\omega}\\ \psi^X_{\hat\omega}\end{bmatrix}\\&=\begin{bmatrix}\left(\frac{m+r}{mr}\right)k_\sigma & \left(\frac{m-1}{m}\right)k_B & \left(\frac{r-1}{r}\right)k_A & -2k_X\end{bmatrix}\mathbf{\Gamma}\begin{bmatrix}\sigma^2_\omega\\ \psi^B\\ \psi^A\\ \psi^X\end{bmatrix}\end{aligned}$$

where $\mathbf{\Gamma}$ is defined as above:

$$\mathbf{\Gamma}=\frac{I-1}{IT^2}\begin{bmatrix}T(T-1) & -m(m-1) & -r(r-1) & -2mr\\ -T & r^2+2r+m & r(r-1) & -2r^2\\ -T & m(m-1) & m^2+2m+r & -2m^2\\ -T & -r(m-1) & -m(r-1) & 2mr\end{bmatrix}$$

$\mathbf{\Gamma}$ is a singular matrix and cannot be inverted. However, we can show that, rather than having no solutions, this system is instead overdetermined and there are an infinite number of solutions. We are simply interested in one such solution. To find one set of k_σ, k_B, k_A, and k_X coefficients that solve this system, we iteratively solve for each k coefficient as a function of the remaining k coefficients and then substitute into the subsequent equations. That is, we use the first equation of this system to solve for k_σ as a function of k_B, k_A, and k_X and substitute this expression in place of k_σ in the subsequent equations in the system, and we repeat for the remaining coefficients and

28. This assumes k_σ, k_B, k_A, and k_X are functions of m, r, and I only, and do not themselves depend on σ^2_ω, ψ^B, ψ^A, or ψ^X. We show this to be true below.

29. Note that if $m=1$ (or $r=1$), the corresponding ψ^B and $\psi^B_{\hat\omega}$ (or ψ^A and $\psi^A_{\hat\omega}$) parameters are undefined and no longer enter the system. Similarly, the corresponding row(s) and column(s) are removed from $\mathbf{\Gamma}$.

equations. This iterative procedure yields:

$$
\begin{aligned}
k_\sigma &= \frac{I(m+r)^2 + (I-1)[r(m-1)k_B + m(r-1)k_A - 2mrk_X]}{(I-1)(m+r)(m+r-1)} \\
k_B &= \frac{I(m+r)^2(m-r+mr+r^2) + (I-1)[m^2(r-1)(2-m-r)k_A + 2mr(r-m-mr-r^2)k_X]}{(I-1)r^2(3m+r+mr+r^2-2)} \\
k_A &= \frac{I(m+r)^2 - 2(I-1)mrk_X}{2(I-1)m^2} \\
k_X &= 0
\end{aligned}
$$

We iteratively substitute each coefficient into the expressions for the remaining coefficients. That is, we first substitute this value of k_X into the expressions for the other three coefficients, then substitute k_A, and so on. This yields expressions for k_σ, k_B, k_A, and k_X in terms of m, r, and I.

$$
\begin{aligned}
k_\sigma &= \frac{I(m+r)^2}{2(I-1)mr} \\
k_B &= \frac{I(m+r)^2}{2(I-1)r^2} \\
k_A &= \frac{I(m+r)^2}{2(I-1)m^2} \\
k_X &= 0
\end{aligned}
$$

We can now express the MDE of an experiment as a function of the residual-based parameters:

$$
\begin{aligned}
MDE^{est} = \left(t^J_{1-\kappa} + t^J_{\alpha/2}\right) \Bigg\{ \left(\frac{1}{P(1-P)J}\right) \Bigg[\left(\frac{m+r}{mr}\right) \left(\frac{I(m+r)^2}{2(I-1)mr}\right) \sigma^2_{\hat{\omega}} \\
+ \left(\frac{m-1}{m}\right) \left(\frac{I(m+r)^2}{2(I-1)r^2}\right) \psi^B_{\hat{\omega}} + \left(\frac{r-1}{r}\right) \left(\frac{I(m+r)^2}{2(I-1)m^2}\right) \psi^A_{\hat{\omega}} \Bigg] \Bigg\}^{1/2}
\end{aligned}
$$

Appendix References

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